MODIFIED MASSIVE ARRATIA FLOW AND WASSERSTEIN DIFFUSION

Vitalii Konarovskyi and Max-K. von Renesse

Max Planck Institut für Mathematik in den Naturwissenschaften and Yuriy Fedkovych Chernivtsi National University, Universität Leipzig

Extending previous work [28] by the first author we present a variant of the Arratia flow, which consists of a collection of coalescing Brownian motions starting from every point of the unit interval. The important new feature of the model is that individual particles carry mass which aggregates upon coalescence and which scales the diffusivity of each particle in an inverse proportional way. In this work we relate the induced measure valued process to the Wasserstein diffusion of [48]. First, we present the process as a martingale solution to a SPDE similar to [48]. Second, as our main result we show a Varadhan formula [44] for short times which is governed by the quadratic Wasserstein distance.

1. Introduction and statement of main results.

1.1. *Motivation.* Since its introduction in [34] Otto's formal infinite dimensional Riemannian calculus for optimal transportation has been the inspiration for numerous new results both in pure and applied mathematics, see e.g. [1, 32, 35, 41, 47]. It can be considered a lift of conventional calculus of points to point ensembles resp. spatially continuous mass distributions. It is therefore natural to ask whether this lifting procedure from points to mass configurations has a probabilistic counterpart. The fundamental object of such a theory would need to be an analogue of Brownian motion on the space of probability measures adapted to Otto's Riemannian structure of optimal transportation. In [48] the second author together with Sturm proposed a first candidate of such a measure valued Brownian motion (with drift), calling it *Wasserstein Diffusion*, and showed among other things that its short time asymptotics are indeed governed by the geometry of optimal transport in the sense of a Varadhan formula for short times governed by the Wasserstein distance. – However, the construction in [48] has several limitations since it is strictly restricted to diffusing measures on the real line, it brings about additional seemingly non-physical correction/renormalization terms and lastly it is obtained by abstract Dirichlet form methods which e.g. do not allow for generic

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starting points of the evolution. Hence, in spite of several ad-hoc finite dimensional approximations [3, 40, 42] the process remained a rather obscure object. Given the strong similarity of the SPDE representation of the Wasserstein Diffusion to the Dean-Kawasaki equation in physics [9, 26] it is natural to ask for related measure valued diffusion processes which share a similar multiplicative noise structure, giving rise to the same large deviation principles on short time scales.

1.2. *Modified Massive Arratia Flow.* In this paper we give a different and very explicit construction of another diffusion process in the space of probability measures on the real line which exhibits a similarity to the Wasserstein diffusion as discussed above. The construction is based on a modification of the so-called Arratia flow of coalescing Brownian motions, which was introduced in [4] and which was later extensively studied by Dorogovtsev and coauthors [14, 15, 16, 17, 33] resp. Le Jan-Raimond [31]. As an important extension the Brownian Web [20] has also received significant attention in recent studies.

Our point of departure is another modification of the Arratia flow in [27] by assigning a mass to each particle, which is aggregated when particles coalesce and which controls the diffusivity of each particle in inverse proportional way. In [28] it was shown for the first time that such a system can be constructed starting with an infinitesimal mass particle at each point of the unit interval, i.e. such that the empirical measure of the particles almost surely converges in weak topology to the uniform measure on the unit interval as time tends to zero. – The resulting model, which we shall call *modified massive Arratia flow (MMAF)*, can best be described in terms of a family of continuous martingales that describe the motion of the particles. Letting $D([0,1],C[0,T])$ denote the Skorokhod space of càdlág-functions from $[0,1]$ into the metric space $(C[0,T],d_\infty)$ of continuous real valued trajectories over the time interval $[0,T]$ with the uniform distance d_∞ and λ denote Lebesgue measure on $[0,1]$, the main result of [28] reads as follows.

THEOREM 1.1. *There is a process $y \in D([0,1],C[0,T])$ such that*

(C1) for all $u \in [0,1]$ the process $y(u,\cdot)$ is a continuous square integrable martingale with respect to the filtration

$$\mathscr{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \le t), \quad t \in [0,T];$$

(C2) for all $u \in [0,1]$, $y(u,0) = u$;
(C3) for all $u < v$ from $[0,1]$ and $t \in [0,T]$, $y(u,t) \le y(v,t)$;
(C4) for all $u,v \in [0,1]$, the joint quadratic variation of $y(u,\cdot)$ and $y(v,\cdot)$ is

$$[y(u,\cdot),y(v,\cdot)]_t = \int_0^t \frac{\mathbb{I}_{\{\tau_{u,v} \le s\}} ds}{m(u,s)},$$

where $m(u,t) = \lambda\{v : \exists s \le t \quad y(v,s) = y(u,s)\}$, $\tau_{u,v} = \inf\{t : \; y(u,t) = y(v,t)\} \wedge T$.

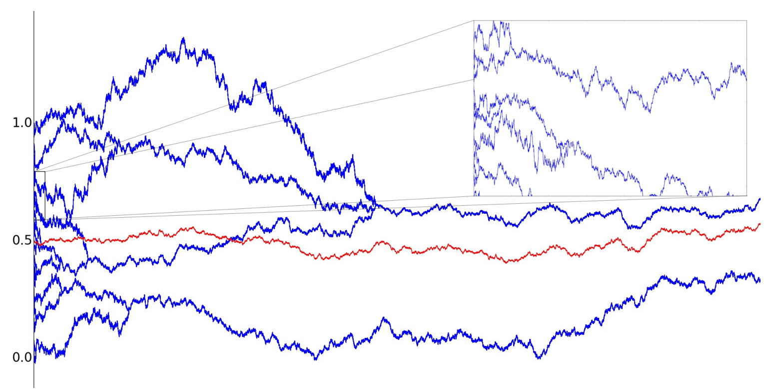
Note that uniqueness in law of y satisfying properties $(C1)-(C4)$ remains an important open problem. However, all subsequent results derived in this paper deal just with *some* field of martingales satisfying properties $(C1)-(C4)$ above. In particular, uniqueness is not needed for any of our arguments.

We also point out that, in contrast to the classical Arratia flow, the family of maps $\{y(\cdot,s)\}_{s\ge 0}$ does not induce a (stochastic) flow on the real line, i.e. does not satisfy a cocycle property. Our terminology of a 'modified massive Arratia flow' refers rather to the corresponding measure valued process[1]

$$\mu_t := y(\cdot,t)_{\#}\lambda, \quad t \in [0,T],$$

which is obtained via the image (push forward) of the uniform measure λ on $[0,1]$ under the random maps $y(\cdot,t)$. The process μ_t, $t \in [0,T]$, is the central object of our interest. In particular, $\mu_0 = \lambda$ in the present case, but our arguments and constructions below can be modified to the case of more general starting measure, cf. [30]. For the sake of presentation, in the sequel we stick to the $\mu_0 = \lambda$ case.

For illustration and comparison to the standard Arratia flow we include here some numerical simulations. The red trajectory on the picture is the evolution of the center of mass of the particles which is a Brownian motion.



1.3. *Main results for the Modfied Massive Arratia Flow.*

1.3.1. *New construction and stochastic calculus for the MMAF.* The first result of this paper is a new simplified construction of a modified massive Arratia flow, using spatial discretization and a tightness argument. Second, we analyze the process $y(t) = y(\cdot,t)$, $t \in [0,T]$, as an $L_2(\lambda)$-valued martingale and develop an associated

[1] In fact μ_t, $t \in [0,T]$, turns out to be a Markov process, but we will not stress this here.

stochastic calculus. We show that for each $g \in L_2(\lambda)$, $s \mapsto J(g)(s) := (g, y(s))_{L_2(\lambda)}$, where $(\cdot,\cdot)_{L_2(\lambda)}$ denotes the inner product in $L_2(\lambda)$, is a continuous square integrable martingale with quadratic variation process

$$[J(g)]_t = \int_0^t \|\mathrm{pr}_{y(s)} g\|^2_{L_2(\lambda)} ds, \quad t \in [0,T].$$

Here $\mathrm{pr}_h g$ is the orthogonal projection in $L_2(\lambda)$ of g onto the subspace of $\sigma(h)$-measurable functions. This shows that the process $y(\cdot)$ is a martingale solution to the infinite dimensional SDE

$$dy(s) = \mathrm{pr}_{y(s)} dW_s,$$

where W is cylindrical Brownian motion in the Hilbert space $L_2(\lambda)$.

By $(C3)$ the map $[0,1] \ni u \to y(u,t)$ is monotone (and càdlág), hence the one-to-one map between probability measures on $\mathbb{R}$ and their quantile functions on $[0,1]$ yields an equivalent parametrization of y by the induced measure valued flow

$$\mu_t := y(\cdot,t)_\# \lambda, \quad t \in [0,T], \tag{1.1}$$

where $y(\cdot,t)_\#\lambda(A) = \lambda\{u : \; y(u,t) \in A\}$, $A \in \mathscr{B}(\mathbb{R})$, denotes the image measure of λ under the map $y(\cdot,t)$.

The process μ_t, $t \in [0,T]$, and its relation to the Wasserstein diffusion, is our main interest of this paper. Our first observation follows from the Ito formula for $y(t)$, $t \in [0,T]$, obtained in [28].

PROPOSITION 1.2. *Let $\mu_t := y(\cdot,t)_\#\lambda$. Then, for each twice continuously differentiable function f on $\mathbb{R}$ with bounded derivatives up to the second order*

$$M_t^f := \langle f, \mu_t \rangle - \int_0^t \langle f, \Gamma(\mu_s) \rangle \, ds$$

is a continuous local martingale with quadratic variation process

$$\left[M^f\right]_t = \int_0^t \langle (f')^2, \mu_s \rangle ds,$$

where Γ is defined as follows

$$\langle f, \Gamma(\nu) \rangle = \frac{1}{2} \sum_{x \in \mathrm{supp}(\nu)} f''(x).$$

We point out that $\Gamma(\mu_t)$ is well defined since property $(P4)$ of section 2.4 below implies, that $\text{supp}(\mu_t)$ is a finite set for all $t \in (0,T]$ almost surely.

As a consequence of Proposition 1.2, μ_t, $t \in [0,T]$, is a probability valued martingale solution to the SPDE

$$d\mu_t = \Gamma(\mu_t)dt + \text{div}(\sqrt{\mu_t}dW_t), \tag{1.2}$$

which follows from a standard application of Ito's formula in finite dimensions. In fact, assuming existence of a solution in $L_2(\lambda)$ of

$$d\mu_t = L(\mu_t)\,dt + \sigma(\mu_t)dW_t$$

with dW being cylindrical $L_2(\lambda)$-white noise, for fixed $f \in C_0^\infty(0,1)$ one deduces from Ito's formula for the smooth maps $F,G : L_2(\lambda) \mapsto \mathbb{R}$ $F(\mu) := \langle f,\mu\rangle$ and $G(\mu) := F^2(\mu)$ that

$$M_t^f := \langle f,\mu_t\rangle - \int_0^t \langle L^* f,\mu_s\rangle ds$$

is a real valued (local) martingale with quadratic variation process

$$[M^f]_t = \int_0^t (\sigma^*(\mu_s)f, \sigma^*(\mu_s)f)_{L_2(\lambda)} ds,$$

where in our case $L(\mu) = \Gamma(\mu)$ and $\sigma(\mu)(\xi) = \text{div}(\sqrt{\mu}\xi)$ such that $\sigma^*(\mu)f = \sqrt{\mu}\cdot\nabla f \in L_2(\lambda)$, i.e.

$$(\sigma^*(\mu)f, \sigma^*(\mu)f)_{L_2(\lambda)} = \langle |\nabla f|^2, \mu\rangle.$$

The SPDE (1.2) above should be compared to the corresponding SPDE for the Wasserstein diffusion [3, 48][2] reading

$$d\mu_t = \beta\Delta\mu_t dt + \hat{\Gamma}(\mu_t)dt + \text{div}(\sqrt{\mu_t}dW_t),$$

with

$$\langle f, \hat{\Gamma}(\nu)\rangle = \sum_{I\in \text{gaps}(\nu)} \left[\frac{f''(I_+) + f''(I-)}{2} - \frac{f'(I_+) - f'(I_+)}{|I|}\right].$$

Thus, besides the apparent similarity of the second order part in the drift operators Γ and $\hat{\Gamma}$, both models share the same singular multiplicative noise which gives rise to the characteristic density $\langle (f')^2, \mu\rangle$ in the quadratic variation process. Of course, this is the same expression as the one appearing in Otto's definition [34] of the Riemannian energy of an infinitesimal (tangential) perturbation of a measure resp. in the Benamou-Brenier formula [5] for optimal transportation.

[2] see also [1] for the connection to the Dean-Kawasaki equation (c.f. [9]).

1.3.2. *Varadhan Formula for the short time asymptotics of the MMAF.* As the main achievement of the present paper we will make this connection more rigorous by showing that the small time fluctuations of the process are in fact governed, on an exponential scale, by the Wasserstein metric.

To this aim, recall that the (quadratic) Wasserstein metric is defined as follows. For probability measures $\nu_1,\ \nu_2$ on the real line with finite second moments it is defined by

$$d_{\mathscr{W}}(\nu_1,\nu_2)=\left(\inf_{\nu\in\chi(\nu_1,\nu_2)}\iint_{\mathbb{R}^2}|\xi-\eta|^2\nu(d\xi,d\eta)\right)^{\frac{1}{2}},$$

where $\chi(\nu_1,\nu_2)$ denotes the set of all probability measures on $\mathbb{R}^2$ with marginals ν_1,ν_2. The main result of the present paper is the following version of the Varadhan formula for the measure valued diffusion μ_t, $t\in[0,T]$. – The precise conditions for a set $A\subset\mathscr{P}(\mathbb{R})$ to be properly chosen for the statement are specified in section 1.3.4 below.

THEOREM 1.3. *Let y satisfy $(C1)-(C4)$ and let μ_t, $t\in[0,T]$, be defined by (1.1). Then, for properly chosen sets $A\subset\mathscr{P}(\mathbb{R})$*

$$\lim_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{\mu_\varepsilon\in A\}=-\frac{(d_{\mathscr{W}}(\lambda,A))^2}{2}, \tag{1.3}$$

where the uniform distribution λ on $[0,1]$ is considered as an element of $\mathscr{P}(\mathbb{R})$.

It should be noted that in Theorem 1.3 we do not make any assumptions on the system y other than $(C1)-(C4)$. Here, the particular construction leading to a system with these properties does not play any role.

Theorem 1.3 is a large deviations statement for the family of random measures μ_ε, involving the rate function

$$I(\eta)=\frac{1}{2}\left(\inf_{\nu\in\chi(\lambda,\eta)}\iint_{\mathbb{R}^2}|\xi-\eta|^2\nu(d\xi,d\eta)\right)^2=\frac{1}{2}d^2_{\mathscr{W}}(\lambda,\eta).$$

We obtain it by contraction from a full large deviation principle for the family of processes $\{y^\varepsilon(\cdot)\}_{\varepsilon\in(0,1]}=\{y(\varepsilon\cdot)\}_{\varepsilon\in(0,1]}$. The latter is the main technical achievement of the present paper and it consumes the biggest part of it.

1.3.3. *Large Deviation Principle for the MMAF.* For a precise statement of the large deviation principle for the sequence $\{y^\varepsilon(\cdot)\}_{\varepsilon\in(0,1]}=\{y(\varepsilon\cdot)\}_{\varepsilon\in(0,1]}$ we need some notation. Let $L_2(\rho) = L_2([0,1],\rho)$, $\rho(du) = \kappa(u)du$, where $\kappa : [0,1] \to [0,1]$

$$\kappa(u) = \begin{cases} u^\beta, & u \in [0,1/2], \\ (1-u)^\beta, & u \in (1/2,1], \end{cases} \quad s \tag{1.4}$$

for some fixed $\beta > 1$, and

$$D^\uparrow = \{h \in D([0,1],\mathbb{R}) : \ h \text{ is non-decreasing}\}.$$

Denote

$$\begin{aligned} \mathscr{H} = \{\varphi \in C([0,T], L_2(\lambda) \cap D^\uparrow) : \ \varphi(0) = \mathrm{id} \text{ and} \\ t \to \varphi(t) \in L_2(\lambda) \text{ is absolutely continuous}^3\}, \end{aligned} \tag{1.5}$$

$$\mathrm{I}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot\varphi(t)\|^2_{L_2(\lambda)}dt, & \varphi \in \mathscr{H}, \\ +\infty, & \text{otherwise.} \end{cases}$$

THEOREM 1.4. *The family of processes* $\{y^\varepsilon\}_{\varepsilon\in(0,1]}$ *satisfies a large deviations principle in the space* $C([0,T],L_2(\rho))$*, endowed with the uniform metric, with the good rate function* I*, i.e. for any open set* G *in* $C([0,T],L_2(\rho))$

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in G\} \geq -\inf_G \mathrm{I}$$

and for any closed set F

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in F\} \leq -\inf_F \mathrm{I}.$$

Since the processes $y^\varepsilon(\cdot)$ solve

$$dy^\varepsilon(s) = \mathrm{pr}_{y^\varepsilon(s)}\sqrt{\varepsilon}dW_s,$$

Theorem 1.4 appears as an instance of the classical Freidlin-Wentzel LDP for solutions of SDE, but here we have to deal with additional difficulties since the diffusion operator $g \to \sigma(g) = \mathrm{pr}_g$ is not continuous as an operator-valued map on $L_2(\lambda)$, and generally little is known about such large deviation principles for solutions of a SDE with non-smooth coefficients even in finite dimensions. In our case we can overcome these difficulties with additional arguments, using the fact that σ is continuous on strictly monotone $y \in L_2(\lambda)$.

[3] A function $f(t)$, $t \in [0,T]$, taking values in a Hilbert space H is called *absolutely continuous* if there exists an integrable function $t \to h(t) \in H$ (in Bochner sense) such that

$$f(t) = f(0) + \int_0^t h(s)ds,$$

and we will denote the function h by $\dot f$.

1.3.4. *Properly chosen subsets $A \subset \mathscr{P}(\mathbb{R})$.* In order to specify the conditions on the set A for the validity of formula (1.3), let τ_ρ denote the image topology on $\mathscr{P}(\mathbb{R})$ of the $L_2(\rho)$-topology on $D^\uparrow([0,1])$ induced from the bijection

$$\iota : g \mapsto g_\# \lambda.$$

We call a set $A \subset \mathscr{P}(\mathbb{R})$ displacement convex if it is the image of a convex subset of $D^\uparrow([0,1])$ under the map ι. A set is properly chosen for the validity of Varadhan's formula as in Theorem 1.3, for instance, if it is displacement convex τ_ρ-closed with non-empty τ_ρ-interior.

REMARK 1.5. It is possible to construct a process y in a similar fashion on a circle S with a proper notion of martingale on S. In this case the family $\{y^\varepsilon(\cdot) = y(\varepsilon\cdot)\}_{\varepsilon\in(0,1]}$ will be exponentially tight in $C([0,T],L_2(\lambda))$, since the state space $L_2^\uparrow(\lambda)$ is compact. Consequently, the large deviation principle can be proved in $C([0,T],L_2(\lambda))$ and thus, it will imply that the Varadhan formula (1.3) holds for any measurable set A that belongs to the space $\mathscr{P}(S)$ of probability measures on S and satisfies $\overline{\mathrm{int} A} = \overline{A}$, for instance.

The organization of the paper is as follows. In section 2 we give a streamlined review of the construction of the modified massive Arratia flow from [28][4]. In section 3 we introduce some elements of a stochastic calculus relative to y to the extent needed in the sequel. The final section 4 is devoted to the proof of the large deviations principle Theorem 1.4.

2. Construction by a system of coalescing heavy diffusion particles.

2.1. *A finite number of particles.* We consider a finite system of particles which start from the points $\frac{k}{n}$, $k = 1,\ldots,n$, with the mass $\frac{1}{n}$, where $n \in \mathbb{N}$ is fixed.

PROPOSITION 2.1. *For each n, there exists a set of processes $\{x_k^n(t),\ k = 1,\ldots, n,\ t \in [0,T]\}$ that satisfies the following conditions*

(F1) for each k, x_k^n is a continuous square integrable martingale with respect to the filtration
$$\mathscr{F}_t^n = \sigma(x_l^n(s),\ s \le t,\ l = 1,\ldots,n);$$
(F2) for all k, $x_k^n(0) = \frac{k}{n}$;
(F3) for all $k < l$ and $t \in [0,T]$, $x_k^n(t) \le x_l^n(t)$;

[4]Here we construct the process directly on $[0,T]$ as a limit of particle systems, whereas in [28] the construction also included an $\varepsilon \to 0$ limit for a sequence of processes on $[\varepsilon,T]$.

(F4) for all k and l,

$$[x_k^n, x_l^n]_t = \int_0^t \frac{\mathbb{I}_{\{\tau_{k,l}^n \le s\}} ds}{m_k^n(s)},$$

where $m_k^n(t) = \frac{1}{n}\#\{j:\ \exists s \le t\ x_j^n(s) = x_k^n(s)\}$, $\tau_{k,l}^n = \inf\{t:\ x_k^n(t) = x_l^n(t)\} \wedge T$ *and* $\#A$ *denotes the number of points of* A.

Such a system of processes can be constructed from a family of independent Wiener processes, coalescing their trajectories. Moreover, $(F1)-(F4)$ uniquely determined the distribution of $x^n = (x_1^n, \ldots, x_n^n)$ in $(C[0,T])^n$ (see [27]).

2.2. *Tightness of a finite system in the space* $D([0,1], C[0,T])$. Let

$$y_n(u,t) = \begin{cases} x_{\lfloor un \rfloor + 1}^n(t), & u \in [0,1), \\ x_n^n(t), & u = 1, \end{cases} \quad t \in [0,T].$$

PROPOSITION 2.2. *The sequence* $\{y_n(u,t),\ u \in [0,1],\ t \in [0,T]\}$ *is tight in* $D([0,1], C[0,T])$.

The statement will follow from theorems 3.8.6 and 3.8.8 [19] and Remark 3.8.9 ibid. The following lemmas 2.3, 2.4 and 2.5 can be used to check conditions (8.39), (8.30) of [19] and (a) of Theorem 3.7.2 ibid., respectively.

LEMMA 2.3. *For all* $n \in \mathbb{N}$, $u \in [0,2]$, $h \in [0,u]$ *and* $\lambda > 0$

$$\mathbb{P}\{d_\infty(y_n(u+h,\cdot), y_n(u,\cdot)) > \lambda,\ d_\infty(y_n(u,\cdot), y_n(u-h,\cdot)) > \lambda\} \le \frac{4h^2}{\lambda^2}.$$

Here $y_n(u,\cdot) = y_n(1,\cdot)$, $u \in [1,2]$, *and* d_∞ *is the uniform distance on* $[0,T]$.

LEMMA 2.4. *For all* $\beta > 1$

$$\lim_{\delta \to 0} \sup_{n \ge 1} \mathbb{E}\left[d_\infty(y_n(\delta,\cdot), y_n(0,\cdot))^\beta \wedge 1\right] = 0.$$

Lemmas 2.3 and 2.4 ware proved in [28] (see lemmas 2.2 and 2.3). The following statement is a new result.

LEMMA 2.5. *For all* $u \in [0,1]$ *the sequence* $\{y_n(u,t),\ t \in [0,T]\}_{n \ge 1}$ *is tight in* $C[0,T]$.

PROOF. To prove the lemma we use the Aldous tightness criterion (see e.g. Theorem 3.6.5. [8]), namely we show that

(A1) for all $t \in [0,T]$ the sequence $\{y_n(u,t)\}_{n\ge 1}$ is tight in $\mathbb{R}$;

(A2) for all $r>0$ each set of stopping times $\{\sigma_n\}_{n\ge 1}$ taking values in $[0,T]$ and each sequence $\delta_n \searrow 0$

$$\lim_{n\to\infty} \mathbb{P}\{|y_n(u,\sigma_n+\delta_n)-y_n(u,\sigma_n)|\ge r\}=0.$$

Note that $(A1)$ follows from Chebyshev's inequality and the estimate

$$\begin{aligned}\mathbb{E}|y_n(u,t)| &\le \mathbb{E}\left|y_n(u,t)-\int_0^1 y_n(q,t)dq\right| + \mathbb{E}\left|\int_0^1 y_n(q,t)dq\right| \\ &\le \mathbb{E}(y_n(1,t)-y_n(0,t)) + \mathbb{E}\left|\int_0^1 y_n(q,t)dq\right| \\ &= 1+\mathbb{E}\left|\int_0^1 y_n(q,t)dq\right|,\end{aligned}$$

where $\int_0^1 y_n(q,t)dq$ is a Wiener process.

Condition $(A2)$ can be checked as follows. Similarly as in the proof of Lemma 2.16 [28], we have that for each $\alpha \in \left(0,\frac{3}{2}\right)$ there exists a constant C such that for all $u\in[0,1]$ and $n\ge 1$

$$\mathbb{E}\frac{1}{m_n^\alpha(u,t)} \le \frac{C}{\sqrt{t}},$$

where

$$m_n(u,t)=\begin{cases} m^n_{[un]+1}(t), & u\in[0,1),\\ m^n_n(t), & u=1,\end{cases} \qquad t\in[0,T].$$

Thus, one can estimate

$$\begin{aligned}&\varlimsup_{n\to\infty} \mathbb{P}\{|y_n(u,\sigma_n+\delta_n)-y_n(u,\sigma_n)|\ge r\}\\ &\qquad\le \frac{1}{r^2}\varlimsup_{n\to\infty}\mathbb{E}(y_n(u,\sigma_n+\delta_n)-y_n(u,\sigma_n))^2\\ &\qquad= \frac{1}{r^2}\varlimsup_{n\to\infty}\mathbb{E}\int_{\sigma_n}^{\sigma_n+\delta_n}\frac{1}{m_n(u,s)}ds\\ &\qquad= \frac{1}{r^2}\varlimsup_{n\to\infty}\mathbb{E}\int_0^T \mathbb{I}_{(\sigma_n,\sigma_n+\delta_n]}\frac{1}{m_n(u,s)}ds\\ &\qquad\le \frac{1}{r^2}\varlimsup_{n\to\infty}\left(\mathbb{E}\int_0^T \mathbb{I}_{(\sigma_n,\sigma_n+\delta_n]}ds\right)^{\frac{1}{4}}\left(\mathbb{E}\int_0^T\frac{1}{m_n^{\frac{4}{3}}(u,s)}ds\right)^{\frac{3}{4}}\\ &\qquad\le \frac{2^{\frac{3}{4}}CT^{\frac{3}{8}}}{r^2}\varlimsup_{n\to\infty}\delta_n^{\frac{1}{4}}=0.\end{aligned}$$

□

2.3. *Martingale characterization of limit points (proof of Theorem 1.1).* Since the space $D([0,1],C[0,T])$ is Polish, the tightness implies the relative compactness of $\{y_n(u,t),\ u\in[0,1],\ t\in[0,T]\}$ in $D([0,1],C[0,T])$. In this section we explain how one can prove that every limit point of $\{y_n\}$ satisfies $(C1)-(C4)$, which proves Theorem 1.1. The idea is the same as in [28].

Let $\{y_{n'}\}$ converge to y weakly in the space $D([0,1],C[0,T])$ for some subsequence $\{n'\}$. By Skorokhod's theorem (see Theorem 3.1.8 [19]) we may suppose that $\{y_{n'}\}$ converge to y in $D([0,1],C[0,T])$ a.s. For convenience of notation we will suppose that the $\{y_n\}$ converge to y in $D([0,1],C[0,T])$ a.s. Next, to prove the theorem, first we show that $y_n(u,\cdot)$ tends to $y(u,\cdot)$ in $C[0,T]$ a.s. Note that, in general, this does not follow from convergence in the space $D([0,1],C[0,T])$. So we need some continuity property of $y(u,\cdot)$, $u\in[0,1]$, in u.

LEMMA 2.6. *For all $u\in[0,1]$ one has*

$$\mathbb{P}\{y(u,\cdot)\neq y(u-,\cdot)\}=0.$$

PROOF. The proof is similar to one of Lemma 2.9 [28]. □

COROLLARY 2.7. *For all $u\in[0,1]$*

$$y_n(u,\cdot)\to y(u,\cdot)\ \text{ in }\ C[0,T]\ \text{ a.s.}$$

Corollary 2.7 and Proposition 9.1.17 [23] immediately imply properties $(C1)-(C3)$. Property $(C4)$ can be proved by the following lemma and the representation of $m(u,t)$ and $m_n(u,t)$ via $\tau_{u,v}$, $v\in[0,1]$, and $\tau^n_{u,v}$, $v\in[0,1]$, i.e.

$$m(u,t)=\int_0^1 \mathbb{I}_{\{\tau_{u,v}\le t\}}dv,$$
$$m_n(u,t)=\int_0^1 \mathbb{I}_{\{\tau^n_{u,v}\le t\}}dv,$$

similarly as it was done in the proofs of lemmas 2.13 and 2.15 [28].

LEMMA 2.8. *Let $\{z_n(t),\ t\in[0,T]\}_{n\ge1}$, be a sequence of continuous local martingales (not necessary with respect to the same filtration) such that for all $n\ge1$ and $s,t\in[0,\tau_n]$, $s<t$*

$$[z_n(\cdot)]_t-[z_n(\cdot)]_s\ge p(t-s), \tag{2.1}$$

where $\tau_n=\inf\{t:\ z_n(t)=0\}\wedge T$ and p is a non-random positive constant. Let $z(t)$, $t\in[0,T]$, be a continuous process such that

$$z(\cdot\wedge\tau)=\lim_{n\to\infty}z_n(\cdot\wedge\tau_n)\ (\text{in }\ C([0,T],\mathbb{R}))\ \text{a.s.},$$

where $\tau = \inf\{t : z(t) = 0\} \wedge T$. *Then*

$$\tau = \lim_{n\to\infty} \tau_n \text{ in probability}. \tag{2.2}$$

PROOF. The proof of this technical lemma can be found in [28, Lemma 2.10]. $\square$

2.4. *Some properties of the modified massive Arratia flow.* Let y satisfy $(C1)-(C4)$. Then the following properties hold.

(P1) For each $\alpha \in \left(0, \frac{3}{2}\right)$ the exists a constant C such that for all $u \in [0,1]$

$$\mathbb{E}\frac{1}{m^{\alpha}(u,t)} \leq \frac{C}{\sqrt{t}}, \quad t \in (0,T].$$

(P2) There exists a constant C such that for all $u \in [0,1]$

$$\mathbb{E}\int_0^t \frac{ds}{m(u,s)} \leq C\sqrt{t}, \quad t \in [0,T].$$

(P3) There exists a constant C such that for all $u \in [0,1]$

$$\mathbb{E}(y(u,t)-u)^2 \leq C\sqrt{t}, \quad t \in [0,T].$$

(P4) For all $t \in (0,T]$ the function $y(u,t)$, $u \in [0,1]$, is a step function in $D([0,1],\mathbb{R})$ with a finite number of jumps. Moreover,

$$\begin{aligned} \mathbb{P}\{\forall u,v \in [0,1], t \in [0,T),\ y(u,t) = y(v,t) \text{ implies} \\ y(u,t+\cdot) = y(v,t+\cdot)\} = 1. \end{aligned} \tag{2.3}$$

REMARK 2.9. According to $(P4)$, hereafter we will suppose that for all $\omega \in \Omega$ and $t \in [0,T)$, $y(\cdot,t,\omega)$ is a step function in $D([0,1],\mathbb{R})$ with a finite number of jumps. Also we assume that for all $u,v \in [0,1]$, $\omega \in \Omega$ and $t \in [0,T)$, $y(u,t,\omega) = y(v,t,\omega)$ implies $y(u,t+\cdot,\omega) = y(v,t+\cdot,\omega)$.

Here $(P1)$ is the statement of Lemma 2.16 [28], $(P2)$ immediately follows from $(P1)$. Property $(P3)$ follows from $(P2)$ and $(C4)$.

PROOF OF $(P4)$. We set

$$\begin{aligned} \Omega' = &\{\forall\, u,v \in [0,1]\cap\mathbb{Q},\ t \in [0,T),\ y(u,t) = y(v,t) \\ &\quad \text{implies } \ y(u,t+\cdot) = y(v,t+\cdot)\} \\ &\cap \left\{\forall n \in \mathbb{N} \ \int_0^1 \frac{du}{m(u,t_n)} < \infty, \text{ where } t_n = \frac{1}{n}\wedge T\right\}. \end{aligned}$$

Since the set $[0,1]\cap\mathbb{Q}$ is countable, Proposition 2.3.4 [36] and $(P1)$ imply $\mathbb{P}\{\Omega'\}=1$.

Next we prove that

$$\text{for every } \omega\in\Omega',\ u\in[0,1]\cap\mathbb{Q},\ v\in[0,u) \text{ and } t\in[0,T)$$
$$y(u,t,\omega)=y(v,t,\omega) \text{ implies } y(u,t+\cdot,\omega)=y(v,t+\cdot,\omega). \tag{2.4}$$

Indeed, if $y(u,t,\omega)=y(v,t,\omega)$, then by the monotonicity of $y(\cdot,t,\omega)$ (see $(C3)$), $y(u,t,\omega)=y(\widetilde{v},t,\omega)$ for all $\widetilde{v}\in[v,u)\cap\mathbb{Q}$. Hence $y(u,t+s,\omega)=y(\widetilde{v},t+s,\omega)$ for all $s\in[0,T-t]$. Using the right-continuity of $y(\cdot,t,\omega)$, we have $y(u,t+s,\omega)=y(v,t+s,\omega)$. This proves (2.4).

Let $\omega\in\Omega'$, $u\in[0,1]$, $v\in[0,u)$ and $t\in[0,T)$ be fixed and let $y(u,t,\omega)=y(v,t,\omega)$. If we show that there exists $\widetilde{u}\in[u,1]\cap\mathbb{Q}$ satisfying $y(\widetilde{u},t,\omega)=y(u,t,\omega)$, then (2.4) will immediately imply (2.3). To check this, we will use the fact that $\int_0^1\frac{d\widehat{u}}{m(\widehat{u},t_n,\omega)}$ is finite for all $n\in\mathbb{N}$.

We fix some element $\widetilde{t}$ from $\{t_n,\ n\in\mathbb{N}\}$ such that $\widetilde{t}\le t$ and assume that for all $\widetilde{u}\in(u,1]\cap\mathbb{Q}$ $\ y(\widetilde{u},t,\omega)>y(u,t,\omega)$. Then the right-continuity of $y(\cdot,t,\omega)$ and its monotonicity imply that there exists a sequence $\{u_n\}_{n\ge1}$ strongly decreasing to u such that $y(u_{n+1},t,\omega)<y(u_n,t,\omega)$ for all $n\in\mathbb{N}$. Next, we set

$$\widetilde{u}_n=\inf\{u':\ y(u',t,\omega)=y(u_n,t,\omega)\},\quad n\in\mathbb{N}.$$

Since $y(\cdot,t,\omega)$ is right-continuous, we have $y(\widetilde{u}_n,t,\omega)=y(u_n,t,\omega)$. Moreover, $\{\widetilde{u}_n\}_{n\ge1}$ also strongly decreases to u and $y(\widetilde{u}_{n+1},t,\omega)<y(\widetilde{u}_n,t,\omega)$ for all $n\in\mathbb{N}$. Consequently, for all $u'\in(\widetilde{u}_{n+1},\widetilde{u}_n)\cap\mathbb{Q}$ and $u''\in(\widetilde{u}_{n+2},\widetilde{u}_{n+1})\cap\mathbb{Q}$, $n\in\mathbb{N}$, $y(u'',t,\omega)<y(u',t,\omega)$, by the monotonicity of $y(\cdot,t,\omega)$ and the choice of the sequence $\{\widetilde{u}_n\}_{n\ge1}$. Thus, $y(u'',r,\omega)<y(u',r,\omega)$ also for each $r\in[0,t]$, since u', u'' are rational and ω was taken from Ω'. Now we can estimate for every $\widehat{u}\in(\widetilde{u}_{n+1},\widetilde{u}_n)$, $n\in\mathbb{N}$,

$$m(\widehat{u},\widetilde{t},\omega)=\lambda\{u':\ \exists r\le\widetilde{t}\ \ y(u',r,\omega)=y(\widehat{u},r,\omega)\}\le\widetilde{u}_n-\widetilde{u}_{n+1}.$$

So,

$$\int_0^1\frac{d\widehat{u}}{m(\widehat{u},\widetilde{t},\omega)}\ge\sum_{n=1}^{\infty}\int_{\widetilde{u}_{n+1}}^{\widetilde{u}_n}\frac{d\widehat{u}}{m(\widehat{u},\widetilde{t},\omega)}\ge\sum_{n=1}^{\infty}\int_{\widetilde{u}_{n+1}}^{\widetilde{u}_n}\frac{d\widehat{u}}{\widetilde{u}_n-\widetilde{u}_{n+1}}=+\infty.$$

But this contradicts the finiteness of the integral $\int_0^1\frac{d\widehat{u}}{m(\widehat{u},\widetilde{t},\omega)}$. Consequently (2.3) holds.

Next, let $t\in(0,T]$ be fixed. We are going to show that $y(\cdot,t)$ is a step function with a finite number of jumps a.s. Let $N(t)$ be a number of distinct points of $B_t=\{y(u,t),\ u\in[0,1]\}$ (that can be equal $+\infty$, if B_t has infinitely many points). Then under (2.3) one can see that

$$N(t) = \int_0^1 \frac{du}{m(u,t)} \quad \text{a.s.} \tag{2.5}$$

Indeed, let for fixed ω, that we omit in the notation, $\pi(u,t) = \{v : \; y(v,t) = y(u,t)\}$, $u \in [0,1]$. Then by (2.3), we have $m(u,t) = \lambda(\pi(u,t))$. Consequently, (2.5) holds, if $N(t)$ is finite. Next, we suppose that $N(t) = +\infty$ and set $A_t = \{u : \; m(u,t) > 0\}$. Note that $\int_0^1 \frac{du}{m(u,t)} = +\infty$ is enough to check only for the case $\lambda(A_t) = 1$. So, assuming that $\lambda(A_t) = 1$ and using the fact that the number of distinct points of B_t is infinite and $y(\cdot,t)$ is non-decreasing, it is easily seen that there exists a set of $\{u_k, \; k \in \mathbb{N}\} \subset [0,1]$ such that $y(u_k,t) \neq y(u_l,t)$ for all $k \neq l$ and $m(u_k,t) > 0$, $k \in \mathbb{N}$. Now, we can estimate

$$\int_0^1 \frac{du}{m(u,t)} \geq \sum_{k=1}^n \int_{\pi(u_k)} \frac{du}{m(u,t)} = \sum_{k=1}^n \int_{\pi(u_k)} \frac{du}{\lambda(\pi(u_k))} = n.$$

Letting $n \to \infty$, we get (2.5).

Thus, $N(t)$ must be finite a.s., by $(P1)$.

Also we would like to note here that (2.3) yields that almost surely for all $t \in (0,T]$ $y(\cdot,t)$ is a step function with a finite number of jumps.

$\square$

3. Some elements of stochastic analysis for the system of heavy diffusion particles. In this section L_2 will denote the space of square integrable measurable functions on $[0,1]$ with respect to Lebesgue measure and $\|\cdot\|_{L_2}$ the usual norm in L_2.

3.1. *Predictable L_2-valued processes.* We will construct the stochastic integral with respect to the flow of particles in the standard way. First we introduce it for simple functions and then we pass to the limit. So in this section we show that each predictable L_2-valued process can be approximated by simple processes. We need some characterization of the Borel σ-field on L_2.

LEMMA 3.1. *The Borel σ-algebra $\mathscr{B}(L_2)$ coincides with the σ-algebra generated by functionals*

$$(\cdot,a), \quad a \in L_2,$$

where $(\cdot,\cdot)$ denotes the inner product on L_2.

PROOF. The assertion follows from Proposition 1.1.1 [7]. $\square$

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a fixed probability space with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$.

DEFINITION 3.2. The σ-algebra $\mathscr{S}$ of subsets of $[0,T]\times\Omega$ generated by sets of the form,

$$(s,t]\times F, \quad 0\le s<t\le T,\ F\in\mathscr{F}_s \quad \text{and} \quad \{0\}\times F, \quad F\in\mathscr{F}_0, \tag{3.1}$$

is called the *predictable* σ-field.

Note that $\mathscr{S}$ is generated by left-continuous adapted processes with values in $\mathbb{R}$ (see Lemma 25.1 [25]).

DEFINITION 3.3. An L_2-valued process $X(t)$, $t\in[0,T]$, is called *predictable* if the map

$$X:[0,T]\times\Omega\to L_2$$

is $\mathscr{S}/\mathscr{B}(L_2)$-measurable.

Lemma 3.1 immediately implies the following assertion.

LEMMA 3.4. *An L_2-valued process $X(t)$, $t\in[0,T]$, is predictable if and only if for each $a\in L_2$ the process $(X(t),a)$, $t\in[0,T]$, is predictable.*

Next we prove a proposition that is similar to Lemma 25.1 [25], namely, $\mathscr{S}$ is generated by L_2-valued processes which are continuous in some sense. For this purpose we introduce the following definition.

DEFINITION 3.5. An L_2-valued process $X(t), t\in[0,T]$, is *weakly left-continuous* if $(X(t),a)$, $t\in[0,T]$, is left-continuous for all $a\in L_2$.

PROPOSITION 3.6. *The σ-algebra $\mathscr{S}$ coincides with the σ-algebra generated by all weakly left-continuous L_2-valued processes.*

PROOF. Let $\mathscr{S}'$ denotes the σ-algebra generated by all weakly left-continuous L_2-valued processes. Lemma 3.1 implies that $\mathscr{S}'$ coincides with the sigma algebra generated by processes of the form,

$$(X(t),a), \quad t\in[0,T],$$

where $a\in L_2$ and X is a weakly left-continuous L_2-valued process. From Lemma 25.1 [25] it follows that $\mathscr{S}'\subseteq\mathscr{S}$.

To proof the inclusion $\mathscr{S}'\supseteq\mathscr{S}$, it is enough to note that every left-continuous process with values in $\mathbb{R}$ is also weakly left-continuous. The proposition is proved. $\square$

For an L_2-valued process $X(t)$, $t \in [0,T]$, set

$$\|X\|_{\mathscr{A}^2} = \left(\mathbb{E} \int_0^T \|X(t)\|_{L_2}^2 dt \right)^{\frac{1}{2}}.$$

Let $\mathscr{A}^2$ denote the set of all predictable L_2-valued processes X with

$$\|X\|_{\mathscr{A}^2} < \infty. \tag{3.2}$$

The map $\|\cdot\|_{\mathscr{A}^2}$ is a norm on $\mathscr{A}^2$.

Consider an L_2-valued process Y of the form

$$Y(t) = \varphi_0 \mathbb{I}_{\{0\}}(t) + \sum_{k=0}^{m-1} \varphi_k \mathbb{I}_{(t_k,t_{k+1}]}(t), \quad t \in [0,T], \tag{3.3}$$

where $0 = t_0 < t_1 < \ldots < t_m = T$ and φ_k, $k = 0,\ldots,m-1$, are L_2-valued random elements. It is easy to see that $Y \in \mathscr{A}^2$ if and only if φ_k is $\mathscr{F}_{t_k}$-measurable and $\mathbb{E}\|\varphi_k\|_{L_2}^2 < \infty$ for all $k = 0,\ldots,m-1$. Denote by $\mathscr{A}_0^2$ the subset of $\mathscr{A}^2$ that contains all processes of the form (3.3).

PROPOSITION 3.7. *The set $\mathscr{A}_0^2$ is dense in $\mathscr{A}^2$.*

PROOF. The proposition can be proved similarly as Proposition 1.4.22 [7]. First, one can show that each $X \in \mathscr{A}^2$ can be approximated in $\mathscr{A}^2$ by a sequence of predictable L_2-valued processes which take only finite numbers of values. Then it is sufficient to check that for an arbitrary $A \in \mathscr{S}$ and all $\varepsilon > 0$ there exists a finite union Γ of disjoint sets of the form (3.1) such that

$$\mathbb{P} \otimes \lambda((A \setminus \Gamma) \cup (\Gamma \setminus A)) < \varepsilon,$$

that follows from the definition of $\mathscr{S}$. This proves the proposition. $\square$

3.2. *The stochastic integral on $\mathscr{A}_0^2$.* In the present section we construct the stochastic integral with respect to the system of coalescing heavy diffusion particles for simple functions (from $\mathscr{A}_0^2$) and prove that it is a square integrable martingale. Consider a random element in $D([0,1],C[0,T])$ which satisfies $(C1)-(C4)$. Hereafter we will assume that $\mathscr{S}$ is the predictable σ-algebra associated with the filtration

$$\mathscr{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \le t), \quad t \in [0,T].$$

Let f belong to $\mathscr{A}_0^2$. Then f can be written in the form (3.3). We set

$$I_t(f) = \int_0^t \int_0^1 f(u,s)dy(u,s)du := \sum_{k=0}^{m} (\varphi_k, y(t_{k+1}\wedge t) - y(t_k \wedge t)) =$$
$$= \sum_{k=0}^{m} \int_0^1 \varphi_k(u)(y(u,t_{k+1}\wedge t) - y(u,t_k\wedge t))du, \quad t\in[0,T].$$

REMARK 3.8. Since $y(t) = y(\cdot,t)$, $t\in[0,T]$, is an L_2-valued random process, usually we will use the notation $I_t(f) = \int_0^t (f(s), dy(s))$.

For $a,b\in L_2$ denote the projection of a onto the space of $\sigma(b)$-measurable functions from L_2 by $\mathrm{pr}_b a$.

The following theorem shows that the total fluctuation of the martingale field y is given by the sizes of level sets of $y(s)$ (see formula (3.5) below).

THEOREM 3.9. *For each $f\in\mathscr{A}_0^2$, $I(f)$ is a continuous square integrable $(\mathscr{F}_t)$-martingale with the quadratic variation*

$$[I(f)]_t = \int_0^t \|\mathrm{pr}_{y(s)} f(s)\|_{L_2}^2 ds, \quad t\in[0,T].$$

To prove the theorem we will use integration in a Banach spaces. So, let H be a Banach space, $\xi:\Omega\to H$ be a random element such that $\mathbb{E}\|\xi\|_H<\infty$ and let $\mathscr{R}$ be a σ-algebra contained in $\mathscr{F}$. Then there exists a unique $\mathscr{R}$-measurable random element η in H such that $\mathbb{E}\|\eta\|_H \le \mathbb{E}\|\xi\|_H$ and for all $A\in\mathscr{R}$, $h^*\in H^*$

$$\int_A (\xi,h^*)d\mathbb{P} = \int_A (\eta,h^*)d\mathbb{P}.$$

This element η is called a conditional expectation of ξ with respect to $\mathscr{R}$ and we will denote it by $\mathbb{E}^{\mathscr{R}}\xi$. More detailed information about conditional expectation for H-valued random elements can be found in [43, Section 2.4.1]. The following statement can also be found in Section 2.4.1 [43] (see Property (f) there). We would like to note that the statement easily follows from the definition of the conditional expectation.

LEMMA 3.10. *If L is a bounded linear operator from H to a Banach space K, then*

$$\mathbb{E}^{\mathscr{R}} L\xi = L\mathbb{E}^{\mathscr{R}}\xi.$$

Let us state the following lemma that is used to prove Theorem 3.9. Let λ_d denote the Lebesgue measure on $[0,1]^d$ for $d\in\mathbb{N}$.

LEMMA 3.11. *Let $d \in \mathbb{N}$ and let $\xi(u)$, $\eta(u)$, $u \in [0,1]^d$, be measurable (in (u,ω)) random fields such that $\mathbb{E}\int_{[0,1]^d}|\xi(u)|du < \infty$, $\eta(u) = \mathbb{E}(\xi(u)|\mathscr{R})$ for almost all $u \in [0,1]^d$ (w.r.t. λ_d), and for each $a \in L_\infty([0,1]^d,\lambda_d)$ let $\int_{[0,1]^d} a(u)\eta(u)du$ be an $\mathscr{R}$-measurable random variable. Then ξ, η are random elements in $L_1([0,1]^d,\lambda_d)$ and $\mathbb{E}^{\mathscr{R}}\xi = \eta$.*

PROOF. The measurability of ξ, η as maps from Ω to $L_1([0,1]^d,\lambda_d)$ follows from the measurability of the fields $\xi(u)$, $\eta(u)$, $u \in [0,1]^d$, and Proposition 1.1.1 [7]. Moreover, η is $\mathscr{R}$-measurable as a map to $L_1([0,1]^d,\lambda_d)$, since $\int_{[0,1]^d} a(u)\eta(u)du$ is an $\mathscr{R}$-measurable random variable for all $a \in L_\infty([0,1]^d,\lambda_d)$. Next, we take $A \in \mathscr{R}$, $a \in L_\infty([0,1]^d,\lambda_d)$ and consider

$$\int_A (a,\eta)d\mathbb{P} = \int_A \left(\int_{[0,1]^d} a(u)\eta(u)du\right)d\mathbb{P} = \int_{[0,1]^d}\left(\int_A a(u)\eta(u)d\mathbb{P}\right)du$$
$$= \int_{[0,1]^d}\left(\int_A a(u)\xi(u)d\mathbb{P}\right)du = \int_A (a,\xi)d\mathbb{P},$$

where $(\cdot,\cdot)$ denotes the inner product in $L_2([0,1]^d,\lambda_d)$. The lemma is proved. □

LEMMA 3.12. *For all $0 \le s < t \le T$ and an arbitrary $\mathscr{F}_s$-measurable L_2-valued random element φ with $\mathbb{E}\|\varphi\|^2_{L_2} < \infty$,*

$$\mathbb{E}\left(\int_0^1\int_0^1 \varphi(u)\varphi(v)A(u,v,t,s)dudv\middle|\mathscr{F}_s\right) = 0.$$

where

$$A(u,v,t,s) = (y(u,t)-y(u,s))(y(v,t)-y(v,s)) - \int_s^t \frac{\mathbb{I}_{\{\tau_{u,v}\le r\}}}{m(u,r)}dr.$$

PROOF. By Lemma 3.10 and Property (d) [43, P. 127], we have

$$\mathbb{E}\left(\int_0^1\int_0^1 \varphi(u)\varphi(v)A(u,v,t,s)dudv\middle|\mathscr{F}_s\right)$$
$$= \int_0^1\int_0^1 \varphi(u)\varphi(v)\left(\mathbb{E}^{\mathscr{F}_s}A(\cdot,t,s)\right)(u,v)dudv.$$

Next, we note that $\mathbb{E}\int_0^1\int_0^1|A(u,v,t,s)|dudv < \infty$, by the Cauchy-Schwarz inequality and Property $(P3)$, and for all $u,v \in [0,1]$

$$\mathbb{E}(A(u,v,t,s)|\mathscr{F}_s) = 0,$$

by $(C4)$. Consequently, Lemma 3.11 yields $\mathbb{E}^{\mathscr{F}_s}A(\cdot,t,s) = 0$, which proves the lemma.

□

LEMMA 3.13. *Under the assumptions of Lemma 3.12,*

$$\int_0^1 \int_0^1 \varphi(u)\varphi(v) \left[\int_s^t \frac{\mathbb{I}_{\{\tau_{u,v} \le r\}}}{m(u,r)} dr\right] du dv = \int_s^t \|\mathrm{pr}_{y(r)}\varphi\|_{L_2}^2 dr. \tag{3.4}$$

PROOF. By Fubini's theorem, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \varphi(u)\varphi(v) \left[\int_s^t \frac{\mathbb{I}_{\{\tau_{u,v} \le r\}}}{m(u,r)} dr\right] du dv \\
&\quad = \int_s^t \left[\int_0^1 \int_0^1 \varphi(u)\varphi(v) \frac{\mathbb{I}_{\{\tau_{u,v} \le r\}}}{m(u,r)} du dv\right] dr \\
&\quad = \int_s^t \left[\int_0^1 \frac{\varphi(u)}{m(u,r)} \left(\int_0^1 \varphi(v)\mathbb{I}_{\{\tau_{u,v} \le r\}} dv\right) du\right] dr \\
&\quad = \int_s^t \left[\int_0^1 \frac{\varphi(u)}{m(u,r)} \left(\int_{\pi(u,r)} \varphi(v) dv\right) du\right] dr,
\end{aligned}$$

where $\pi(u,t) = \{v : \ y(v,t) = y(u,t)\}$. Here we have used the equality $\pi(u,t) = \{v : \ \tau_{u,v} \le r\}$, which follows from Remark 2.9. We note that for each ω and r the operator $\mathrm{pr}_{y(r,\omega)}$ is a usual projection (in $L_2(\lambda)$) onto the subspace of all $\sigma(y(r,\omega))$-measurable functions and moreover

$$\left(\mathrm{pr}_{y(r,\omega)}\varphi(\cdot,\omega)\right)(u) = \frac{1}{m(u,r,\omega)} \int_{\pi(u,r,\omega)} \varphi(v,\omega) dv \tag{3.5}$$

because $y(r,\omega)$ is a step function according to Remark 2.9. Consequently, the left hand side of (3.4) equals

$$\int_s^t \left[\int_0^1 \varphi(u) \left(\mathrm{pr}_{y(r)}\varphi\right)(u) du\right] dr = \int_s^t \|\mathrm{pr}_{y(r)}\varphi\|_{L_2}^2 dr.$$

The lemma is proved. $\square$

PROOF OF THEOREM 3.9. Let $f \in \mathscr{A}_0^2$. The continuity of $I(f)$ follows from its construction. Next, the martingale property of the integral will be checked. Fix $s < t$ and note that f on $(s,t]$ has the form

$$f(r) = \sum_{k=0}^{p-1} \varphi_k \mathbb{I}_{(s_k, s_{k+1}]}(r), \quad r \in (s,t],$$

where $s = s_0 < s_1 < \ldots < s_p = t$ and φ_k is $\mathscr{F}_{s_k}$-measurable, $k = 0, \ldots, p-1$. Using lemmas 3.10, 3.11 and $(C1)$, we calculate

$$\begin{aligned}\mathbb{E}\left(I_t(f)-I_s(f)\middle|\mathscr{F}_s\right)&=\mathbb{E}\left(\sum_{k=0}^{p-1}(\varphi_k,y(s_{k+1})-y(s_k))\middle|\mathscr{F}_s\right)\\&=\sum_{k=0}^{p-1}\mathbb{E}\left(\int_0^1\varphi_k(u)(y(u,s_{k+1})-y(u,s_k))du\middle|\mathscr{F}_s\right)\\&=\sum_{k=0}^{p-1}\int_0^1\mathbb{E}^{\mathscr{F}_s}(\varphi_k\cdot(y(s_{k+1})-y(s_k)))(u)du\\&=\sum_{k=0}^{p-1}\int_0^1\mathbb{E}^{\mathscr{F}_s}\left[\mathbb{E}^{\mathscr{F}_{s_k}}(\varphi_k\cdot(y(s_{k+1})-y(s_k)))\right](u)du\\&=\sum_{k=0}^{p-1}\int_0^1\mathbb{E}^{\mathscr{F}_s}\left[\varphi_k\mathbb{E}^{\mathscr{F}_{s_k}}(y(s_{k+1})-y(s_k))\right](u)du=0.\end{aligned}$$

Therefore $I(f)$ is a martingale. It should be noted that $I(f)$ is a square integrable martingale. This follows from $\mathbb{E}\left(\int_0^1 y^2(u,t)\right)du<\infty$ (see Lemma 2.18 [28] or $(P3)$).

Let us calculate the quadratic variation of $I(f)$. Denote

$$M(t)=I_t^2(f)-\int_0^t\|\mathrm{pr}_{y(s)}f(s)\|_{L_2}^2ds,\quad t\in[0,T].$$

So,

$$\begin{aligned}\mathbb{E}\left(M(t)-M(s)\middle|\mathscr{F}_s\right)=\mathbb{E}&\left((I_t(f)-I_s(f))^2\middle|\mathscr{F}_s\right)\\&-\mathbb{E}\left(\int_s^t\|\mathrm{pr}_{y(r)}f(r)\|_{L_2}^2dr\middle|\mathscr{F}_s\right)\\&+2I_s(f)\mathbb{E}\left((I_t(f)-I_s(f))\middle|\mathscr{F}_s\right).\end{aligned}$$

The third term in the left hand side of the latter relation equals 0 since $I(f)$ is a martingale. Next we calculate

$$\begin{aligned}
&\mathbb{E}((I_t(f)-I_s(f))^2|\mathscr{F}_s)\\
&\qquad = \mathbb{E}\left[\left(\sum_{k=0}^{p-1}\int_0^1 \varphi_k(u)(y(u,s_{k+1})-y(u,s_k))du\right)^2\Bigg|\mathscr{F}_s\right]\\
&\qquad = \sum_{k,l=0}^{p-1}\mathbb{E}\left[\left(\int_0^1 \varphi_k(u)(y(u,s_{k+1})-y(u,s_k))du\right)\right.\\
&\qquad\quad \left.\cdot\left(\int_0^1 \varphi_l(u)(y(u,s_{l+1})-y(u,s_l))du\right)\Bigg|\mathscr{F}_s\right]\\
&\qquad = \sum_{k=0}^{p-1}\mathbb{E}\left[\left(\int_0^1 \varphi_k(u)(y(u,s_{k+1})-y(u,s_k))du\right)^2\Bigg|\mathscr{F}_s\right].
\end{aligned}$$

Here, for $l>k$ we have used the equality

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^1 \varphi_k(u)(y(u,s_{k+1})-y(u,s_k))du\right)\right.\\
&\qquad \left.\cdot\left(\int_0^1 \varphi_l(u)(y(u,s_{l+1})-y(u,s_l))du\right)\Bigg|\mathscr{F}_s\right]\\
&\quad = \mathbb{E}\left[\int_0^1 \varphi_k(u)(y(u,s_{k+1})-y(u,s_k))du\right.\\
&\qquad \left.\cdot\,\mathbb{E}\left\{\int_0^1 \varphi_l(u)(y(u,s_{l+1})-y(u,s_l))du\Bigg|\mathscr{F}_{s_l}\right\}\Bigg|\mathscr{F}_s\right]=0.
\end{aligned}$$

Thus, using lemmas 3.12 and 3.13, it is easily seen that

$$\mathbb{E}\left((I_t(f)-I_s(f))^2\big|\mathscr{F}_s\right)=\mathbb{E}\left[\int_s^t \|\mathrm{pr}_{y(r)}f(r)\|_{L_2}^2dr\Bigg|\mathscr{F}_s\right].$$

The theorem is proved. □

3.3. *Stochastic integral for predictable functions.* Let $\mathscr{M}_2$ denote the space of continuous square integrable $(\mathscr{F}_t)$-martingales $M(t)$, $t\in[0,T]$ with the norm

$$\|M\|_{\mathscr{M}_2}=\left(\mathbb{E}M^2(T)\right)^{\frac{1}{2}}.$$

It is well-known that $\mathscr{M}_2$ is a complete and separable metric space (see e.g. Lemma 2.1.2 [22] for the completeness).

In the previous section we have proved that $I : \mathscr{A}_0^2 \to \mathscr{M}_2$ is a linear operator. Moreover,

$$\|I(f)\|_{\mathscr{M}_2}^2 = \mathbb{E}\left(\int_0^T \int_0^1 f(u,t)dy(u,t)\right)^2 = \mathbb{E}\int_0^T \|\mathrm{pr}_{y(t)} f(t)\|_{L_2}^2 dt \\ \leq \mathbb{E}\int_0^T \|f(t)\|_{L_2}^2 dt = \|f\|_{\mathscr{A}^2}^2. \tag{3.6}$$

Hence I is bounded on $\mathscr{A}_0^2$ and consequently it can be extended to the bounded operator on $\mathscr{A}^2$. We will denote this extension by $\int_0^{\cdot}\int_0^1 f(u,t)dy(u,t)$ or $\int_0^{\cdot}(f(t),dy(t))$.

PROPOSITION 3.14. *For all $f \in \mathscr{A}^2$*

$$\left[\int_0^{\cdot}(f(s),dy(s))\right]_t = \int_0^t \|\mathrm{pr}_{y(s)} f(s)\|_{L_2}^2 ds, \quad t \in [0,T]. \tag{3.7}$$

PROOF. The proof of the proposition, it is enough to check that

$$\mathbb{E}\left[\int_s^t (f(r),dy(r))\middle|\mathscr{F}_s\right] = 0$$

and

$$\mathbb{E}\left[\left(\int_s^t (f(r),dy(r))\right)^2\middle|\mathscr{F}_s\right] = \mathbb{E}\left[\int_s^t \|\mathrm{pr}_{y(r)} f(r)\|_{L_2}^2 dr\middle|\mathscr{F}_s\right],$$

whenever $0 \leq s \leq t \leq T$. But these relations follow easily from the approximation of f by elements of $\mathscr{A}_0^2$ and Theorem 3.9. □

Let $\mathscr{A}$ denotes the set of predictable L_2-valued processes $f(t)$, $t \in [0,T]$, such that

$$\int_0^T \|f(t)\|_{L_2}^2 dt < \infty \quad \text{a.s.} \tag{3.8}$$

Then similarly as in [22, P. 52] one can construct the stochastic integral $\int_0^{\cdot}(f(s),dy(s))$ for all $f \in \mathscr{A}$. Moreover such an integral is a continuous local square integrable martingale with respect to $(\mathscr{F}_t)$ with the quadratic variation as in (3.7).

3.4. *Girsanov's theorem.* In this section we construct a system of coalescing diffusion particles with drift that will be needed in Section 4.2.2 for the proof of the lower bound in LDP. So, fix $\varphi \in \mathscr{A}$ and consider on $(\Omega,\mathscr{F})$ the new measure

$$\mathbb{P}^{\varphi}(A) = \mathbb{E}\mathbb{I}_A \exp\left\{\int_0^T (\varphi(s),dy(s)) - \frac{1}{2}\int_0^T \|\mathrm{pr}_{y(t)}\varphi(t)\|_{L_2}^2 dt\right\}, \quad A \in \mathscr{F}.$$

If

$$\mathbb{E}\exp\left\{\int_0^T(\varphi(s),dy(s))-\frac{1}{2}\int_0^T\|\mathrm{pr}_{y(t)}\varphi(t)\|_{L_2}^2dt\right\}=1, \tag{3.9}$$

then $\mathbb{P}^{\varphi}$ is a probability measure.

THEOREM 3.15. *Let $\varphi\in\mathscr{A}$ satisfy (3.9). Then the random element $\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ in $D([0,1],C[0,T])$ satisfies the following properties under $\mathbb{P}^{\varphi}$*

(D1) for all $u\in[0,1]$ the process

$$\eta(u,\cdot)=y(u,\cdot)-\int_0^{\cdot}\left(\mathrm{pr}_{y(s)}\varphi(s)\right)(u)ds$$

is a continuous local square integrable $(\mathscr{F}_t)$-martingale;
(D2) for all $u\in[0,1]$, $y(u,0)=u$;
(D3) for all $u<v$ from $[0,1]$ and $t\in[0,T]$, $y(u,t)\le y(v,t)$;
(D4) for all $u,v\in[0,1]$ and $t\in[0,T]$,

$$[\eta(u,\cdot),\eta(v,\cdot)]_t=\int_0^t\frac{\mathbb{I}_{\{\tau_{u,v}\le s\}}ds}{m(u,s)}.$$

Note that $(D2)$ and $(D3)$ immediately follows from the absolute continuity of $\mathbb{P}^{\varphi}$. To prove $(D1)$ and $(D4)$ we state an auxiliary lemma.

LEMMA 3.16. *For each $u\in[0,1]$*

$$y(u,t)=u+\int_0^t\int_0^1\frac{\mathbb{I}_{\pi(u,s-)}(q)}{m(u,s-)}dy(q,s)dq.$$

PROOF. Setting $f(q,s)=\frac{\mathbb{I}_{\pi(u,s-)}(q)}{m(u,s-)}$ and using $(P2)$, we have

$$\begin{aligned}\mathbb{E}\int_0^T\|f(s)\|_{L_2}^2ds&=\mathbb{E}\int_0^T\int_0^1\frac{\mathbb{I}_{\pi(u,s-)}(q)}{m^2(u,s-)}dsdq\\&=\mathbb{E}\int_0^T\left(\frac{1}{m^2(u,s)}\int_{\pi(u,s)}dq\right)ds\\&=\mathbb{E}\int_0^T\frac{1}{m(u,s)}ds<\infty.\end{aligned}$$

Next, put

$$\begin{aligned}&\sigma_0=t,\\&\sigma_k=\inf\{s:\ N(s)\le k\}\wedge t,\quad k\in\mathbb{N},\end{aligned}$$

where $N(t)=\int_0^1 \frac{1}{m(u,t)}$, $t\in[0,T]$, denotes a number of distinct points in $\{y(u,t),\ u\in[0,1]\}$ and is an $(\mathscr{F}_t)$-adapted càdlág process, and note that σ_k is an $(\mathscr{F}_t)$-stopping time, $\sigma_k\geq\sigma_{k+1}$. So,

$$\begin{aligned}\int_0^t\int_0^1 f(q,s)dy(q,s)dq &= \sum_{k=0}^{\infty}\int_{\sigma_{k+1}}^{\sigma_k}\int_0^1\frac{\mathbb{I}_{\pi(u,s-)}(q)}{m(u,s-)}dy(q,s)dq\\ &=\sum_{k=0}^{\infty}\int_0^1\frac{\mathbb{I}_{\pi(u,\sigma_{k+1})}(q)}{m(u,\sigma_{k+1})}(y(q,\sigma_k)-y(q,\sigma_{k+1}))dq\\ &=\sum_{k=0}^{\infty}\int_{\pi(u,\sigma_{k+1})}dq\frac{1}{m(u,\sigma_{k+1})}(y(u,\sigma_k)-y(u,\sigma_{k+1}))\\ &=\sum_{k=0}^{\infty}(y(u,\sigma_k)-y(u,\sigma_{k+1}))=y(u,t)-u.\end{aligned}$$

The lemma is proved. □

COROLLARY 3.17. *For each $f\in\mathscr{A}$ and $u\in[0,1]$*

$$\left[\int_0^{\cdot}(f(s),dy(s)),y(u,\cdot)\right]_t=\int_0^t\left(\mathrm{pr}_{y(s)}f(s)\right)(u)ds,\quad t\in[0,T].$$

PROOF OF THEOREM 3.15. The proof of the assertion follows from Girsanov's theorem (see Theorem 5.4.1 [22]) and Corollary 3.17. □

REMARK 3.18. For functions $f\in\mathscr{A}^2$ (resp. $\mathscr{A}$) we can construct the stochastic integral with respect to the flow $\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ satisfying conditions $(D1)-(D4)$ in the same way as in the case of conditions $(C1)-(C4)$. Moreover,

$$\int_0^t(f(s),dy(s))=\int_0^t(f(s),\mathrm{pr}_{y(s)}\varphi(s))ds+\int_0^t(f(s),d\eta(s))$$

and $\int_0^{\cdot}(f(s),d\eta(s))$ is a continuous square integrable (resp. local square integrable) $(\mathscr{F}_t)$-martingale with the quadratic variation

$$\left[\int_0^{\cdot}(f(s),d\eta(s))\right]_t=\int_0^t\|\mathrm{pr}_{y(s)}f(s)\|_{L_2}^2ds.$$

4. Large deviation principle for the modified massive Arratia flow.

4.1. *Exponential tightness.* In this section we prove exponential tightness of the modified massive Arratia flow. In order to prove this we will use “exponentially fast” version of Jakubowski’s tightness criterion (see Theorem A.1 [11]). So, let $\{y(u,t), u \in [0,1], t \in [0,T]\}$ be a random element in $D([0,1], C[0,T])$ satisfying $(C1)-(C4)$ and $\rho(du) = \kappa(u)du$, where κ is given by (1.4).

REMARK 4.1. Since for each $u \in [0,1]$ the process $y(u,t)$, $t \in [0,T]$, has continuous trajectories, $y(t) = y(\cdot,t)$, $t \in [0,T]$, is a continuous $L_2(\rho)$-valued random process, by $(C3)$ and the dominated convergence theorem.

We will establish exponential tightness of $\{y^\varepsilon\}_{\varepsilon\in(0,1]}$ in the space $C([0,T], L_2(\rho))$, where $y^\varepsilon(t) = y(\cdot, \varepsilon t)$, $t \in [0,T]$.

By Theorem A.1 [11], $\{y^\varepsilon\}_{\varepsilon\in(0,1]}$ is exponential tight in $C([0,T], L_2(\rho))$, i.e. for every $M > 0$ there exists a compact $K_M \subset C([0,T], L_2(\rho))$, such that

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \notin K_M\} \le -M,$$

if and only if

(E1) for every $M > 0$ there exists a compact $K_M \subset L_2(\rho)$ such that

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{\exists t \in [0,T] :\ y^\varepsilon(t) \notin K_M\} \le -M; \tag{4.1}$$

(E2) for every $h \in L_2(\rho)$ the sequence $\{(h, y^\varepsilon(\cdot))_{L_2(\rho)}\}_{\varepsilon\in(0,1]}$ is exponentially tight in $C([0,T], \mathbb{R})$, where $(\cdot,\cdot)_{L_2(\rho)}$ denotes the inner product in $L_2(\rho)$.

Since $y(u,t)$, $u \in [0,1]$, is non-decreasing for all $t \in [0,T]$, to find the compact $K_M \subset L_2(\rho)$ satisfying (4.1) it suffices to control the behavior of processes $y(u,t)$, $t \in [0,T]$, for u close to 0 or 1. Note that the diffusion rate of the process $y(u,t)$, $t \in [0,T]$, tends to infinity as $t \to 0$. But

$$M_*(u,t) \le y(u,t) \le M^*(u,t), \quad t \in [0,T], \tag{4.2}$$

where

$$M^*(u,t) = \frac{1}{1-u}\int_u^1 y(v,t)dv,$$
$$M_*(u,t) = \frac{1}{u}\int_0^u y(v,t)dv$$

and

$$\frac{d[M^*(u,\cdot)]_t}{dt} \le \frac{1}{1-u}, \quad \frac{d[M_*(u,\cdot)]_t}{dt} \le \frac{1}{u}. \tag{4.3}$$

The latter inequalities follow from the simple relations

$$M^*(u,t) = \frac{1}{1-u}\int_0^t (\mathbb{I}_{[u,1]}, dy(s)),$$
$$M_*(u,t) = \frac{1}{u}\int_0^t (\mathbb{I}_{[0,u]}, dy(s)),$$

where these sort of integrals ware defined in Section 3.3, and the formula for the quadratic variation of the stochastic integral (3.7). Indeed,

$$\frac{d[M^*(u,\cdot)]_t}{dt} = \frac{1}{(1-u)^2}\|\mathrm{pr}_{y(t)}\mathbb{I}_{[u,1]}\|_{L_2}^2 \le \frac{1}{(1-u)^2}\|\mathbb{I}_{[u,1]}\|_{L_2}^2 = \frac{1}{1-u}.$$

The inequality $\frac{d[M_*(u,\cdot)]_t}{dt} \le \frac{1}{u}$ can be obtained in the same way.

In Lemma 4.3 we will use inequalities (4.2) in order to find a compact $K_M \subset L_2(\rho)$ for each $M > 0$ such that (4.1) holds.

Since $y(u,t)$, $u \in [0,1]$, belongs to $D([0,1],\mathbb{R})$ and is a non-decreasing function, we will often work with non-decreasing functions

$$D^{\uparrow} = \{h \in D([0,1],\mathbb{R}) : \ h \text{ is non-decreasing}\}.$$

LEMMA 4.2. *The set*

$$A_M = \left\{h \in D^{\uparrow} : \ h(1/n) \ge -Mn \text{ and } h(1-1/n) \le Mn, \ n \in \mathbb{N}\right\}.$$

is compact in $L_2(\rho)$ *for all positive* M.

PROOF. First we prove that $A_M \subset L_2(\rho)$. Let $h \in A_M$. Without loss of generality let h be positive on $[1/2,1]$ and negative on $[0,1/2]$. Then

$$\int_0^1 h^2(u)\rho(du) = \int_0^1 h^2(u)\kappa(u)du \le C\sum_{n=2}^{\infty}\frac{M^2n^2}{n^{\beta}}\left(\frac{1}{n-1} - \frac{1}{n}\right) < C_1$$

and C_1 is independent of h.

Next, take a sequence $\{h_k\}_{k\ge1}$ in A_M. Since $\{h_k\}_{k\ge1} \subset D^{\uparrow}$, there exists a subsequence $\{h_{k'}\}$ that convergences to $h \in D^{\uparrow}$ for all $u \in [0,1]$ except possibly countably many points, and hence also ρ-a.e. Since $|h_k(u)| \le f(u)$, $u \in [0,1]$, where

$$f(u) = \begin{cases} Mn, & u \in [1-1/(n-1), 1-1/n), \\ Mn, & u \in [1/n, 1/(n-1)), \end{cases}$$

and $f \in L_2(\rho)$, $\|h_{k'}\|_{L_2(\rho)} \to \|h\|_{L_2(\rho)}$, by the dominated convergence theorem. Consequently, this and Lemma 1.32 [25] imply $h_{k'} \to h$ in $L_2(\rho)$. The lemma is proved. $\square$

LEMMA 4.3. *The family of processes* $\{y^{\varepsilon}\}_{\varepsilon\in(0,1]}$ *satisfies* $(E1)$*, i.e. for every* $M>0$ *there exists a compact* $K_M\subset L_2(\rho)$*, such that* (4.1) *holds.*

PROOF. By Lemma 4.2, we can take $K_M=A_L$ and show that for some $L>0$ (4.1) holds. So,

$$\mathbb{P}\{\exists t\in[0,T]:\ y^{\varepsilon}(t)\notin A_L\}\le\sum_{n=1}^{\infty}\mathbb{P}\{\exists t\in[0,T]:\ y^{\varepsilon}(1/n,t)<-Ln\}\\+\sum_{n=1}^{\infty}\mathbb{P}\{\exists t\in[0,T]:\ y^{\varepsilon}(1-1/n,t)>Ln\}.$$

Using (4.2), we estimate for fixed $n\in\mathbb{N}$

$$\mathbb{P}\{\exists t\in[0,T]:\ y(1-1/n,\varepsilon t)>Ln\}=\mathbb{P}\left\{\sup_{t\in[0,T]}y(1-1/n,\varepsilon t)>Ln\right\}\\\le\mathbb{P}\left\{\sup_{t\in[0,T]}M^*(1-1/n,\varepsilon t)>Ln\right\}$$

Next, by the representation theorem for martingales [22, Theorem 2.7.2'], there exists a Wiener process $w_n(t)$, $t\ge0$, maybe on an extended probability space, such that

$$M^*(1-1/n,t)=M^*(1-1/n,0)+w_n\left([M^*(1-1/n,\cdot)]_t\right),\quad t\in[0,T].$$

Thus,

$$\sup_{t\in[0,T]}M^*(1-1/n,\varepsilon t)\le\sup_{t\in[0,T]}\left(1+w_n\left([M^*(1-1/n,\cdot)]_{\varepsilon t}\right)\right)\\\le\sup_{t\in[0,T]}\left(1+w_n(n\varepsilon t)\right),$$

since the set $\{[M^*(1-1/n,\cdot)]_{\varepsilon t},\ t\in[0,T]\}$ is contained in $\{n\varepsilon t,\ t\in[0,T]\}$, by (4.3).

Hence,

$$\mathbb{P}\left\{\sup_{t\in[0,T]}M^*(1-1/n,\varepsilon t)>Ln\right\}\le\mathbb{P}\left\{\sup_{t\in[0,T]}(w_n(n\varepsilon t)+1)>Ln\right\}\\=\frac{2}{\sqrt{2\pi n\varepsilon T}}\int_{Ln-1}^{\infty}e^{-\frac{x^2}{2n\varepsilon T}}dx\\\le C\exp\left\{-\frac{L^2n}{2\varepsilon T}+\frac{L}{\varepsilon T}\right\},$$

where C is independent of ε, L and n. Here we used the reflection principle for the evaluation of $\mathbb{P}\left\{\sup_{t\in[0,T]}(w_n(n\varepsilon t)+1)>Ln\right\}$ (see e.g. Proposition 2.9 [21]).

Similarly

$$\mathbb{P}\{\exists t\in[0,T]:\ y(1/n,\varepsilon t)<-Ln\}\le C\exp\left\{-\frac{L^2n}{2\varepsilon T}+\frac{L}{\varepsilon T}\right\}.$$

Now, for $M>0$ we can estimate

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\ln\mathbb{P}\{\exists t\in[0,T]:\ y^{\varepsilon}(t)\notin A_L\}&\\ &\le\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\ln\left(2C\sum_{n=1}^{\infty}\exp\left\{-\frac{L^2n}{2\varepsilon T}+\frac{L}{\varepsilon T}\right\}\right)\\ &\le-\frac{L^2}{2T}+\frac{L}{T}<-M,\end{aligned}$$

where L is taken large enough. The lemma is proved. □

LEMMA 4.4. *The sequence of processes $\{y^{\varepsilon}\}_{\varepsilon\in(0,1]}$ satisfies $(E2)$, i.e. for every $h\in L_2(\rho)$ the sequence $\{(h,y^{\varepsilon}(\cdot))_{L_2(\rho)}\}_{\varepsilon\in(0,1]}$ is exponentially tight in $C([0,T],\mathbb{R})$.*

PROOF. To prove the lemma, we will use Corollary 7.1 [38] (see also Theorem 3 [37]). It is enough to show that for each h there exist positive constants α, γ and k such that for all $s,t\in[0,T]$, $s<t$

$$\mathbb{E}\exp\left\{\frac{\gamma}{\varepsilon(t-s)^{\alpha}}|M_h(\varepsilon t)-M_h(\varepsilon s)|\right\}\le k^{1/\varepsilon},\quad\forall\varepsilon\le\varepsilon_0,$$

where $M_h(t)=(h,y(t))_{L_2(\rho)}$, $t\in[0,T]$.

Using (3.7), for

$$M_h(t)=\int_0^1 h(u)y(u,t)\kappa(u)du=\int_0^t(h\kappa,dy(s))_{L_2(\lambda)}$$

we have

$$[M_h]_t=\int_0^t\|\mathrm{pr}_{y(s)}(h\kappa)\|^2_{L_2(\lambda)}ds\le\int_0^t\|h\kappa\|^2_{L_2(\lambda)}ds\le\|h\|^2_{L_2(\rho)}t.$$

The inequality for the quadratic variation of M_h and Novikov's theorem [25, Theorem 18.23] imply

$$\mathbb{E}\exp\left\{\beta\int_s^t(h\kappa,dy(r))_{L_2(\lambda)}-\frac{\beta^2}{2}\int_s^t\|\mathrm{pr}_{y(r)}(h\kappa)\|^2_{L_2(\lambda)}dr\right\}=1.\tag{4.4}$$

So, for $\delta > 0$

$$\begin{aligned}\mathbb{E}\exp\{\delta|M_h(\varepsilon t)-M_h(\varepsilon s)|\} &\le \mathbb{E}\exp\{\delta(M_h(\varepsilon t)-M_h(\varepsilon s))\}\\ &\quad+\mathbb{E}\exp\{\delta(M_h(\varepsilon s)-M_h(\varepsilon t))\}\\ &= \mathbb{E}\exp\left\{\delta\int_{\varepsilon s}^{\varepsilon t}(h\kappa, dy(r))_{L_2(\lambda)} - \frac{\delta^2}{2}\int_{\varepsilon s}^{\varepsilon t}\|\mathrm{pr}_{y(r)}(h\kappa)\|^2_{L_2(\lambda)}dr\right.\\ &\quad\left.+\frac{\delta^2}{2}\int_{\varepsilon s}^{\varepsilon t}\|\mathrm{pr}_{y(r)}(h\kappa)\|^2_{L_2(\lambda)}dr\right\} + \mathbb{E}\exp\{\delta(M_{-h}(\varepsilon t)-M_{-h}(\varepsilon s))\}\\ &\le 2\mathbb{E}\exp\left\{\frac{\varepsilon\delta^2}{2}\|h\|^2_{L_2(\rho)}(t-s)\right\}.\end{aligned}$$

Taking $\delta = \frac{\sqrt{2}}{\varepsilon(t-s)^{1/2}\|h\|_{L_2(\rho)}}$, we have

$$\mathbb{E}\exp\left\{\frac{\sqrt{2}|M_h(\varepsilon t)-M_h(\varepsilon s)|}{\varepsilon\|h\|_{L_2(\rho)}(t-s)^{1/2}}\right\} \le 2e^{1/\varepsilon} \le (2e)^{1/\varepsilon}.$$

This finishes the proof of the lemma. □

From the two previous lemmas we obtain the exponential tightness of $\{y^\varepsilon\}_{\varepsilon\in(0,1]}$.

PROPOSITION 4.5. *The sequence $\{y^\varepsilon\}_{\varepsilon\in(0,1]}$ is exponentially tight in $C([0,T], L_2(\rho))$.*

4.2. *Proof of Theorem 1.4.* We set

$$L_2^\uparrow(\rho) = \{g \in L_2(\rho) : \exists \widetilde{g} \in D^\uparrow,\ g = \widetilde{g},\ \rho\text{-a.e.}\}$$

and

$$C_{\mathrm{id}}([0,T], L_2^\uparrow(\rho)) = \{\varphi \in C([0,T], L_2^\uparrow(\rho)) :\ \varphi(0) = \mathrm{id}\}.$$

REMARK 4.6. Since the set $C_{\mathrm{id}}([0,T], L_2^\uparrow(\rho))$ is closed in $C([0,T], L_2(\rho))$, it is enough to state LDP for $\{y^\varepsilon\}_{\varepsilon\in(0,1]}$ in the metric space $C_{\mathrm{id}}([0,T], L_2^\uparrow(\rho))$.

Due to the exponential tightness, for the upper bound it is enough to consider compact sets. According to [10] (see Theorem 4.1.11), for this it is enough to show that I is a lower-semicontinuous function and

(B1) weak upper bound:

$$\lim_{r\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \le -\mathrm{I}(\varphi),$$

where $\varphi \in C_{\mathrm{id}}([0,T], L_2^\uparrow(\rho))$ and $B_r(\varphi)$ is the open ball in $C_{\mathrm{id}}([0,T], L_2^\uparrow(\rho))$ with center φ and radius r;

(B2) lower bound: for every open set $A \subseteq C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho))$

$$\varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in A\} \geq -\inf_{\varphi \in A} \mathrm{I}(\varphi).$$

To prove the upper and lower bounds we will follow the idea in [12, 13] based on exponential change of measure and the Girsanov transformation.

4.2.1. *The upper bound.* First we check $(B1)$. We set

$$H = \{h \in C([0,T], L_2(\rho^{-1})) : \dot{h} \in L_2([0,T], L_2(\rho^{-1}))\},$$

where $\rho^{-1}(du) = \frac{1}{\kappa(u)} du$.

For $h \in H$ let

$$M_t^{\varepsilon,h} = \exp\left\{\frac{1}{\varepsilon}\left[\int_0^t (h(s), dy^\varepsilon(s))_{L_2(\lambda)} - \frac{1}{2}\int_0^t \|\mathrm{pr}_{y^\varepsilon(s)} h(s)\|_{L_2(\lambda)}^2 ds\right]\right\}. \tag{4.5}$$

By Novikov's theorem, $M_t^{\varepsilon,h}$, $t \in [0,T]$, is a martingale with $\mathbb{E}M_t^{\varepsilon,h} = 1$ (see also (4.4)). By an integration by parts (Lemma A.1), we can write

$$M_T^{\varepsilon,h} = \exp\left\{\frac{1}{\varepsilon} F(y^\varepsilon, h)\right\},$$

where

$$\begin{aligned} F(\varphi, h) = &(h(T), \varphi(T))_{L_2(\lambda)} - (h(0), \mathrm{id})_{L_2(\lambda)} \\ &- \int_0^T (\dot{h}(s), \varphi(s))_{L_2(\lambda)} ds \\ &- \frac{1}{2}\int_0^T \|\mathrm{pr}_{\varphi(s)} h(s)\|_{L_2(\lambda)}^2 ds, \quad \varphi \in C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho)). \end{aligned}$$

For $\varphi \in C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho))$ we have

$$\begin{aligned} \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} &= \mathbb{E}\left[\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}} \frac{M_T^{\varepsilon,h}}{M_T^{\varepsilon,h}}\right] \\ &\leq \exp\left\{-\frac{1}{\varepsilon}\inf_{\psi \in B_r(\varphi)} F(\psi, h)\right\} \mathbb{E}M_T^{\varepsilon,h} \\ &= \exp\left\{-\frac{1}{\varepsilon}\inf_{\psi \in B_r(\varphi)} F(\psi, h)\right\}. \end{aligned}$$

Using the inequality $\|pr_{\varphi(s)} h(s)\|_{L_2(\lambda)}^2 \leq \|h(s)\|_{L_2(\lambda)}^2$, we obtain

$$\varlimsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq -\inf_{\psi \in B_r(\varphi)} F(\psi, h) \leq -\inf_{\psi \in B_r(\varphi)} \Phi(\psi, h),$$

where

$$\Phi(\varphi,h)=(h(T),\varphi(T))_{L_2(\lambda)}-(h(0),\mathrm{id})_{L_2(\lambda)}\\ -\int_0^T(\dot h(s),\varphi(s))_{L_2(\lambda)}ds\\ -\frac{1}{2}\int_0^T\|h(s)\|^2_{L_2(\lambda)}ds,\quad \varphi\in C_{\mathrm{id}}([0,T],L_2^{\uparrow}(\rho)).$$

Since the map $\Phi(\varphi,h)$, $\varphi\in C_{\mathrm{id}}([0,T],L_2^{\uparrow}(\rho))$, is continuous for fixed h,

$$\lim_{r\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\ln\mathbb{P}\{y^{\varepsilon}\in B_r(\varphi)\}\le -\Phi(\varphi,h).$$

Minimizing in $h\in H$, we obtain

$$\lim_{r\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\ln\mathbb{P}\{y^{\varepsilon}\in B_r(\varphi)\}\le -\sup_{h\in H}\Phi(\varphi,h).$$

Now $(B1)$ will follow from the following.

PROPOSITION 4.7. *For each* $\varphi\in C_{\mathrm{id}}([0,T],L_2^{\uparrow}(\rho))$

$$\sup_{h\in H}\Phi(\varphi,h)=\mathrm{I}(\varphi).$$

PROOF. First we prove the assertion of the proposition for φ satisfying

$$J(\varphi):=\sup_{h\in H}\Phi(\varphi,h)<\infty.$$

Replacing h by θh, $\theta\in\mathbb{R}$, and using the linearity of H, we get

$$J(\varphi)=\sup_{h\in H}\Phi(\varphi,\theta h)$$

for all $\theta\in\mathbb{R}$. Next, we note that for each $h\in H$ the function

$$\theta\mapsto\Phi(\varphi,\theta h)=\theta G(\varphi,h)-\frac{\theta^2}{2}\int_0^T\|h(s)\|^2_{L_2(\lambda)}ds,$$

where

$$G(\varphi,h)=(h(T),\varphi(T))_{L_2(\lambda)}-(h(0),\mathrm{id})_{L_2(\lambda)}\\ -\int_0^T(\dot h(s),\varphi(s))_{L_2(\lambda)}ds,$$

reaches its maximum at the point

$$\theta_h^{\max} = \frac{G(\varphi,h)}{\int_0^T \|h(s)\|^2_{L_2(\lambda)} ds}$$

Consequently,

$$J(\varphi) = \sup_{h\in H} \Phi(\varphi,\theta h) = \sup_{h\in H} \Phi(\varphi,\theta_h^{\max} h),$$

which implies

$$J(\varphi) = \frac{1}{2} \sup_{h\in H} \frac{G^2(\varphi,h)}{\int_0^T \|h(s)\|^2_{L_2(\lambda)} ds} < \infty. \tag{4.6}$$

We can consider H as linear subspace of $L_2([0,T],L_2(\lambda))$. By Lemma A.2, it is a dense subspace and consequently the linear form

$$G_\varphi : h \to G(\varphi,h),$$

which is continuous on H, by (4.6), can be extended to the space $L_2([0,T], L_2(\lambda))$. Using the Riesz theorem, there exists a function $k_\varphi \in L_2([0,T], L_2(\lambda))$ such that

$$G(\varphi,h) = \int_0^T (k_\varphi(s),h(s))ds. \tag{4.7}$$

Thus, by Lemma A.3, φ is absolutely continuous and $\dot{\varphi} = k_\varphi$. Applying the Cauchy-Schwarz inequality to (4.7) we get

$$\begin{aligned} G(\varphi,h)^2 &\le \int_0^T \|k_\varphi(s)\|^2_{L_2(\lambda)} ds \cdot \int_0^T \|h(s)\|^2_{L_2(\lambda)} ds \\ &= 2\mathrm{I}(\varphi) \int_0^T \|h(s)\|^2_{L_2(\lambda)} ds, \end{aligned}$$

with equality for h proportional to k_φ. The latter inequality yields $J(\varphi) \le \mathrm{I}(\varphi)$ and since H is dense in $L_2([0,T],L_2(\lambda))$, we get the equality $J(\varphi) = \mathrm{I}(\varphi)$.

If $\mathrm{I}(\varphi) < \infty$, then φ is absolutely continuous and $k_\varphi = \dot{\varphi}$ in (4.7). So, $J(\varphi) \le \mathrm{I}(\varphi) < \infty$ and consequently we have $J(\varphi) = \mathrm{I}(\varphi)$. This completes the proof of the proposition. □

COROLLARY 4.8. I *is lower-semicontinuous as supremum of continuous functions.*

4.2.2. *The lower bound.*

LEMMA 4.9. *Let there exist a subset $\mathscr{R} \subset C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho))$ such that*

(i) for each $\varphi \in \mathscr{R}$

$$\lim_{r\to 0} \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^{\varepsilon} \in B_r(\varphi)\} \geq -\mathrm{I}(\varphi); \tag{4.8}$$

(ii) for each φ satisfying $\mathrm{I}(\varphi) < \infty$, there exists a sequence $\{\varphi_n\} \subset \mathscr{R}$ such that $\varphi_n \to \varphi$ in $C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho))$ and $\mathrm{I}(\varphi_n) \to \mathrm{I}(\varphi)$.

Then the lower bound $(B2)$ holds.

PROOF. First we note that it is enough to prove $(B2)$ for all open $A \subseteq C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho))$ satisfying $\inf_{\varphi\in A} \mathrm{I}(\varphi) < \infty$.

Let δ be an arbitrary positive number. Then there exists $\varphi_0 \in A$ such that

$$\mathrm{I}(\varphi_0) < \inf_{\varphi\in A} \mathrm{I}(\varphi) + \delta.$$

Hence, by (ii) and the openness of A we can find $\varphi_1 \in A \cap \mathscr{R}$ that satisfies

$$\mathrm{I}(\varphi_1) < \mathrm{I}(\varphi_0) + \delta.$$

Next, using (i) and the openness of A, the exists $r > 0$ such that $B_r(\varphi_1) \subseteq A$ and

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^{\varepsilon} \in B_r(\varphi_1)\} \geq -\mathrm{I}(\varphi_1) - \delta.$$

Consequently, we can now estimate

$$\begin{aligned} \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^{\varepsilon} \in A\} &\geq \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^{\varepsilon} \in B_r(\varphi_1)\} \\ &\geq -\mathrm{I}(\varphi_1) - \delta > -\mathrm{I}(\varphi_0) - 2\delta > -\inf_{\varphi\in A} \mathrm{I}(\varphi) - 3\delta. \end{aligned}$$

Making $\delta \to 0$, we obtain $(B2)$. The lemma is proved. $\square$

So, in order to obtain the lower bound $(B2)$, it is enough to find a subset $\mathscr{R} \subset C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\rho))$ that satisfies (i) and (ii) of Lemma 4.9.

We denote

$$D^{\uparrow\uparrow} = \{g \in D([0,1], \mathbb{R}) : \ \forall u < v \in [0,1], \ g(u) < g(v)\} \tag{4.9}$$

and define $L_2^{\uparrow\uparrow}(\rho)$ in the same way as $L_2^{\uparrow}(\rho)$, replacing $D^{\uparrow}$ by $D^{\uparrow\uparrow}$. Set

$$\mathscr{R}=\Big\{\varphi\in C([0,T],L_2^{\uparrow\uparrow}(\lambda)):\ \mathrm{I}(\varphi)<\infty,\ \dot\varphi\in H_{L_2(\lambda)},$$
$$\dot\varphi \text{ is continuous in } (u,t) \text{ and } \varphi(u,t) \text{ is continuously differentiable in } u$$
$$\text{with bounded (uniformly in } t,u\text{) derivative } \frac{\partial\varphi(u,t)}{\partial u}\Big\},$$

where

$$H_{L_2(\lambda)}=\{h\in C([0,T],L_2(\lambda)):\ \dot h\in L_2([0,T],L_2(\lambda))\}.$$

For $h\in H_{L_2(\lambda)}$ we define the new probability measure $\mathbb{P}^{\varepsilon,h}$ with density

$$\frac{d\mathbb{P}^{\varepsilon,h}}{d\mathbb{P}}=M_T^{\varepsilon,h},$$

where $M_T^{\varepsilon,h}$ is defined by (4.5). By Novikov's theorem and Theorem 3.15, the random element y^ε in $D([0,1],C[0,T])$ satisfies (w.r.t. $\mathbb{P}^{\varepsilon,h}$) the following properties

$(D^\varepsilon 1)$ for all $u\in[0,1]$ the process

$$\eta^\varepsilon(u,\cdot)=y^\varepsilon(u,\cdot)-\int_0^\cdot\left(\mathrm{pr}_{y^\varepsilon(s)}h(s)\right)(u)ds$$

is a continuous local square integrable $(\mathscr{F}_{\varepsilon t})$-martingale;

$(D^\varepsilon 2)$ for all $u\in[0,1]$, $y^\varepsilon(u,0)=u$;

$(D^\varepsilon 3)$ for all $u<v$ from $[0,1]$ and $t\in[0,T]$, $y^\varepsilon(u,t)\le y^\varepsilon(v,t)$;

$(D^\varepsilon 4)$ for all $u,v\in[0,1]$ and $t\in[0,T]$,

$$[\eta^\varepsilon(u,\cdot),\eta^\varepsilon(v,\cdot)]_t=\varepsilon\int_0^t\frac{\mathbb{I}_{\{\tau^\varepsilon_{u,v}\le s\}}ds}{m^\varepsilon(u,s)},$$

where τ^ε and m^ε is defined in the same way as τ and m, replacing y by y^ε.

Note that if $\varphi\in\mathscr{R}$, then

$$\lim_{\varepsilon\to 0}\mathbb{P}^{\varepsilon,\dot\varphi}\{y^\varepsilon\in B_r(\varphi)\}=1 \tag{4.10}$$

for all $r>0$, by Proposition B.1.

Let us fix $\varphi\in\mathscr{R}$ and set $h=\dot\varphi$. Noting that $y^\varepsilon\in L_2([0,T],L_2(\lambda))$ a.s., we estimate

$$\begin{aligned}\mathbb{P}\{y^\varepsilon\in B_r(\varphi)\}&=\mathbb{E}^{\varepsilon,h}\frac{\mathbb{I}_{\{y^\varepsilon\in B_r(\varphi)\}}}{M_T^{\varepsilon,h}}\\&\ge\exp\left\{-\frac{1}{\varepsilon}\sup_{\psi\in B_r(\varphi)\cap L_2([0,T],L_2(\lambda))}F(\psi,h)\right\}\mathbb{P}^{\varepsilon,h}\{y^\varepsilon\in B_r(\varphi)\},\end{aligned}$$

where $\mathbb{E}^{\varepsilon,h}$ denotes the expectation w.r.t. $\mathbb{P}^{\varepsilon,h}$. Thus, by (4.10),

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq - \sup_{\psi \in B_r(\varphi)\cap L_2([0,T],L_2(\lambda))} F(\psi,h). \tag{4.11}$$

Next we prove the continuity of the map $g \mapsto \mathrm{pr}_g f$ on $L_2^{\uparrow\uparrow}(\rho)$ for each $f \in L_2(\lambda)$.

LEMMA 4.10. *Let $g \in L_2^{\uparrow\uparrow}(\rho)$ and $f \in L_2(\lambda)$. If a sequence $\{g_n\}_{n\geq 1}$ of elements $L_2^{\uparrow}(\rho)$ converges to g a.e., then $\{\mathrm{pr}_{g_n} f\}_{n\geq 1}$ converges to f in $L_2(\lambda)$ and*

$$\lim_{n\to\infty} \|\mathrm{pr}_{g_n} f\|_{L_2(\lambda)} = \|f\|_{L_2(\lambda)}. \tag{4.12}$$

PROOF. First we note that $\sigma(g)$ is a Borel σ-algebra on $[0,1]$. Moreover, since $\{g_n\}_{n\geq 1}$ converges to g a.e., one can show that for almost all $a,b \in [0,1]$ and $a<b$ there exists a sequence $\{c_n,\ d_n,\ n\in\mathbb{N}\}$ such that $\lambda((a,b) \triangle g_n^{-1}(c_n,d_n)) \to 0$ as $n\to\infty$. This immediately implies that for all $A \in \sigma(g)$ there exist $A_n \in \sigma(g_n)$, $n \in \mathbb{N}$, such that $\lambda(A \triangle A_n) \to 0$. Thus, by Proposition 1 [2], $\mathrm{pr}_{g_n} f \to \mathrm{pr}_g f = f$ in L_2. Consequently, we also have $\|\mathrm{pr}_{g_n} f\|_{L_2(\lambda)} \to \|f\|_{L_2(\lambda)}$. The lemma is proved. $\square$

So, by Lemma 4.10, (4.11) yields

$$\lim_{r\to 0} \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq -F(\varphi,h) = -\mathrm{I}(\varphi).$$

Here the last equality follows from the form of the map F, the choice of h and Lemma A.1.

PROPOSITION 4.11. *For each $\varphi \in \mathscr{H}$, where $\mathscr{H}$ is defined by (1.5), satisfying $\mathrm{I}(\varphi) < \infty$, there exists a sequence $\{\varphi_n\} \subset \mathscr{R}$ such that $\varphi_n \to \varphi$ in $C_{\mathrm{id}}([0,T],L_2^{\uparrow}(\rho))$ and $\mathrm{I}(\varphi_n) \to \mathrm{I}(\varphi)$.*

REMARK 4.12. To prove the proposition, it is enough to check that for all $\varepsilon > 0$ there exists $\psi \in \mathscr{R}$ such that

$$\int_0^T \|\dot\varphi(t) - \dot\psi(t)\|^2_{L_2(\lambda)} dt < \varepsilon. \tag{4.13}$$

REMARK 4.13. From (4.13) it follows that it is enough to check the statement only for functions φ with a bounded derivative, i.e.

$$\sup_{(u,t)\in[0,1]\times[0,T]} |\dot\varphi(u,t)| < \infty. \tag{4.14}$$

For measurable functions f,g from $\mathbb{R}^2$ to $\mathbb{R}$ let $f*g$ denote the convolution of f and g, i.e

$$f*g(u,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(u-v,t-r)g(v,r)dudr, \quad u,t\in\mathbb{R}.$$

PROOF OF PROPOSITION 4.11. Let φ satisfy $\mathrm{I}(\varphi)<\infty$ and (4.14) hold. Set

$$C = \sup_{(u,t)\in[0,1]\times[0,T]} |\dot{\varphi}(u,t)|$$

and extend φ to the function on $\mathbb{R}^2$. Denoting

$$\widetilde{\varphi}(u,t) = \begin{cases} \dot{\varphi}(u,t), & (u,t)\in[0,1]\times[0,T],\\ C, & (u,t)\in(1,2]\times[0,T],\\ -C, & (u,t)\in[-1,0)\times[0,T],\\ 0, & \text{otherwise,}\end{cases}$$

the extension of φ is defined as

$$\varphi(\cdot,t) = \mathrm{id}\cdot\mathbb{I}_{[-1,2]}(\cdot) + \int_{-\infty}^{t}\widetilde{\varphi}(\cdot,s)ds,$$

where $\int_{-\infty}^{t}\widetilde{\varphi}(\cdot,s)ds$ is the Bochner integral in the space of square integrable functions on $\mathbb{R}$ w.r.t. the Lebesgue measure.

Let ς be a positive twice continuously differentiable symmetric function on $\mathbb{R}$ with $\operatorname{supp}\varsigma\in[-1,1]$ and $\int_{\mathbb{R}}\varsigma(u)du=1$. Set for $0<\delta<1$ and $\alpha>0$

$$\varsigma_\delta(u,t) = \frac{1}{\delta^2}\varsigma\left(\frac{u}{\delta}\right)\varsigma\left(\frac{t}{\delta}\right), \quad u,t\in\mathbb{R}.$$

and

$$\psi(u,t) = u + \int_0^t \widetilde{\varphi}*\varsigma_\delta(u,s)ds + \alpha tu, \quad u\in[0,1],\ t\in[0,T].$$

Note that the continuity property of the convolution (see Theorem 11.21 [18]) in the space of square integrable functions implies that (4.13) holds for small enough δ and α. Thus, to prove the proposition, we need to show that ψ belongs to $\mathscr{R}$. Since $\widetilde{\varphi}*\varsigma_\delta$ is smooth because ς is smooth and the function $\widetilde{\varphi}$ is bounded with compact support, the statement will hold if we check that $\psi(\cdot,t)\in D^{\uparrow\uparrow}$ for all $t\in[0,T]$, where $D^{\uparrow\uparrow}$ is defined by (4.9). So, let $f_0(u,t)=u$ for $u\in[-1,2]$, $t\in\mathbb{R}$ and $f_0=0$ for $u\in\mathbb{R}\setminus[-1,2]$, $t\in\mathbb{R}$. Using the symmetry of ς we have

$$u = f_0*\varsigma_\delta(u,t), \quad u\in[0,1],\ t\in\mathbb{R}.$$

Thus we may rewrite the integral

$$\begin{aligned} u+\int_0^t \widetilde{\varphi}*\varsigma_\delta(u,s)ds &= u+\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{\varphi}(u-v,s-r)\varsigma_\delta(v,r)dsdrdv \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left[f_0(u-v)+\int_0^t \widetilde{\varphi}(u-v,s-r)ds\right]\varsigma_\delta(v,r)drdv \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left[f_0(u-v)+\int_0^{t-r} \widetilde{\varphi}(u-v,s)ds\right]\varsigma_\delta(v,r)drdv \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(u-v,t-r)\varsigma_\delta(v,r)drdv. \end{aligned}$$

Since $\varphi(v_1,t)\le\varphi(v_2,t)$ for all $v_1<v_2$ from $[-1,2]$ and $t\in\mathbb{R}$, we obtain

$$u_1+\int_0^t \widetilde{\varphi}*\varsigma_\delta(u_1,s)ds \le u_2+\int_0^t \widetilde{\varphi}*\varsigma_\delta(u_2,s)ds, \quad u_1,u_2\in[0,1],\ u_1<u_2.$$

This proves the proposition, since the constant α in the definition of ψ is strictly positive.

□

PROOF OF THEOREM 1.3. First of all we note that the family $\{y(\varepsilon)\}_{\varepsilon\in(0,T]}$ satisfies large deviations in $L_2(\rho)$ with the rate function $\frac{1}{2}\|\mathrm{id}-\cdot\|^2_{L_2(\lambda)}$ (for simplicity of notation we suppose that $\|\mathrm{id}-g\|_{L_2(\lambda)}=+\infty$ whenever $g\notin L_2(\lambda)$). This immediately follows from Theorem 1.4 and the contraction principle for large deviations. Next, let us show that for each convex closed set C in $L_2(\rho)$ with non-empty interior we have

$$\lim_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{y(\varepsilon)\in C\} = -\frac{1}{2}\inf_{g\in C}\|\mathrm{id}-g\|^2_{L_2(\lambda)}. \tag{4.15}$$

To prove this, it is enough to show that

$$\inf_{g\in C^\circ}\|\mathrm{id}-g\|_{L_2(\lambda)} \le \inf_{g\in C}\|\mathrm{id}-g\|_{L_2(\lambda)}, \tag{4.16}$$

where C° denotes the interior of C in $L_2(\rho)$. Let $\inf_{g\in C}\|\mathrm{id}-g\|_{L_2(\lambda)}<\infty$ and $\delta\in(0,1]$ be fixed. Then there exists $g_0\in L_2(\lambda)\cap C$ such that

$$\|\mathrm{id}-g_0\|_{L_2(\lambda)} \le \inf_{g\in C}\|\mathrm{id}-g\|_{L_2(\lambda)}+\delta.$$

Since C° is non-empty, there exists $g_1\in C^\circ$ and $r>0$ such that $B(g_1,r):=\{g\in L_2(\rho):\ \|g-g_1\|_{L_2(\rho)}<r\}\subseteq C$. Moreover, g_1 can be chosen from $L_2(\lambda)$ because $L_2(\lambda)$ is dense in $L_2(\rho)$. Next, let $g_c=(1-\delta_0)g_0+\delta_0 g_1$, where $\delta_0=\frac{\delta}{1+\|g_0+g_1\|_{L_2(\lambda)}}$.

Using the convexity of C, we can see that $B(g_c, \delta_0 r) \subseteq C$ and consequently, g_c belongs to C°. Now we can estimate

$$\begin{aligned}\inf_{g\in C^\circ} \|\mathrm{id}-g\|_{L_2(\lambda)} &\le \|\mathrm{id}-g_c\|_{L_2(\lambda)} \le \|\mathrm{id}-g_0\|_{L_2(\lambda)} + \|g_0-g_c\|_{L_2(\lambda)} \\ &\le \|\mathrm{id}-g_0\|_{L_2(\lambda)} + \delta_0 \|g_0+g_1\|_{L_2(\lambda)} \\ &< \inf_{g\in C} \|\mathrm{id}-g\|_{L_2(\lambda)} + 2\delta.\end{aligned}$$

Making $\delta \to 0$, we obtain (4.16).

Thus, the Varadhan formula (1.3) is obtained from a straightforward combination (4.15), the contraction principle for large deviations applied to the endpoint map $C([0,1], L_2(\rho)) \to L_2(\rho)$, $\mu_{t\in[0,1]} \to \mu_1$ and the fact that the map

$$\iota : D^{\uparrow}([0,1]) \ni g \mapsto g_{\#}\lambda \in \mathscr{P}(\mathbb{R})$$

is an isometry from the $L_2(\lambda)$-metric to the quadratic Wasserstein metric $d_{\mathscr{W}}$ (see e.g. Section 2.1 [6]). □

APPENDIX A: SOME PROPERTIES OF ABSOLUTELY CONTINUOUS FUNCTIONS

LEMMA A.1. *For every absolutely continuous function $f(t)$, $t \in [0,T]$, with values in $L_2(\lambda)$*

$$\begin{aligned}\int_0^t (f(s), dy(s)) &= (f(t), y(t))_{L_2(\lambda)} - (f(0), y(0))_{L_2(\lambda)} \\ &\quad - \int_0^t (\dot f(s), y(s))_{L_2(\lambda)} ds, \quad t \in [0,T],\end{aligned} \tag{A.1}$$

almost surely, where the integral in the left hand side was defined in Section 3.3.

PROOF. Observe that, since the right and left hand sides of (A.1) are continuous functions almost surely, it is enough to prove that for each fixed $t \in [0,T]$

$$\begin{aligned}\int_0^t (f(s), dy(s)) &= (f(t), y(t))_{L_2(\lambda)} - (f(0), y(0))_{L_2(\lambda)} \\ &\quad - \int_0^t (\dot f(s), y(s))_{L_2(\lambda)} ds \quad \text{a.s.}\end{aligned}$$

Next, let for each $n \in \mathbb{N}$

$$f_n(s) = f(0)\mathbb{I}_{\{0\}}(s) + \sum_{k=1}^n f(t_{k-1})\mathbb{I}_{(t_{k-1},t_k]}(s), \quad s \in [0,t],$$

where $t_k = \frac{kt}{n}$, $k = 0, \ldots, n$. Then by the continuity of f and the dominated convenience theorem,

$$\int_0^t \|f(s) - f_n(s)\|^2_{L_2(\lambda)} ds \to 0 \quad \text{as} \ \ n \to \infty.$$

Consequently, by inequality (3.6) (where T is replaced by t),

$$\begin{aligned}\int_0^t (f(s), dy(s))_{L_2(\lambda)} &= \lim_{n\to\infty} \int_0^t (f_n(s), dy(s))_{L_2(\lambda)} \\ &= \lim_{n\to\infty} \sum_{k=1}^n (f(t_{k-1}), y(t_k) - y(t_{k-1}))_{L_2(\lambda)}\end{aligned}$$

in $L_2(\Omega)$. Thus, the statement follows from the standard argument. □

LEMMA A.2. *The set H, which is defined in Subsection 4.2.1, i.e.*

$$H = \{h \in C([0,T], L_2(\rho^{-1})) : \ \dot{h} \in L_2([0,T], L_2(\rho^{-1}))\},$$

where $\rho^{-1}(du) = \frac{1}{\kappa(u)} du$, is dense in $L_2([0,T], L_2(\lambda))$.

PROOF. To shorten notation, we set $U = L_2(\rho^{-1})$ and $V = L_2(\lambda)$. Let also $\|\cdot\|_U$ and $\|\cdot\|_V$ denote the norms in U and V, respectively.

Since U is continuously embedded in V, that is, there exists a constant C such that

$$\|a\|_V \le C\|a\|_U \quad \text{for all} \ \ a \in U,$$

we have that

$$\|f\|_{L_2(V)} \le C\|f\|_{L_2(U)} \quad \text{for all} \ \ f \in L_2(U),$$

where $\|\cdot\|_{L_2(U)}$ and $\|\cdot\|_{L_2(V)}$ denote the norms in $L_2(U) = L_2([0,T], L_2(\rho^{-1}))$ and $L_2(V) = L_2([0,T], L_2(\lambda))$, respectively. Thus, if a sequence $\{f_n\}_{n\ge1}$ converges to f in $L_2(U)$, it will converge to f in $L_2(V)$. Also, we note that $L_2(U)$ is dense in $L_2(V)$. This follows from the fact that the set of elementary functions

$$\left\{ f = \sum_{k=1}^n a_k \mathbb{I}_{A_k} : \ \ a_k \in U, \ \ A_k \in \mathscr{B}([0,T]), \ \ n \in \mathbb{N} \right\} \subset L_2(U)$$

is dense in $L_2(V)$, by the density of U in V. Consequently, to prove the lemma, it is enough to show that H is dense in $L_2(U)$.

Let $\{e_n\}_{n\ge1}$ denote an orthonormal basis in U and $f \in L_2(U)$, $\varepsilon > 0$ be fixed. We set

$$\widetilde{f}_n(t) = (f(t), e_n)_U, \quad t \in [0,T], \ \ n \in \mathbb{N},$$

where $(\cdot,\cdot)_U$ denotes the inner product in U. It is easily seen that $\widetilde{f}_n \in L_2([0,T],\mathbb{R})$. Moreover, by Parseval's identity and the monotone convergence theorem

$$\|f\|^2_{L_2(U)} = \int_0^T \left\| \sum_{n=1}^{\infty} \widetilde{f}_n(t) e_n \right\|_U^2 dt = \sum_{n=1}^{\infty} \int_0^T \widetilde{f}_n^2(t)dt < \infty.$$

So, we can find $N \in \mathbb{N}$ such that

$$\|f - f_N\|^2_{L_2(U)} = \sum_{n=N+1}^{\infty} \int_0^T \widetilde{f}_n^2(t)dt < \frac{\varepsilon^2}{4},$$

where $f_N = \sum_{n=1}^N \widetilde{f}_n e_n$.

Next, since the space of continuously differentiable functions $C^1([0,T])$ on $[0,T]$ is dense in $L_2([0,T],\mathbb{R})$, for each $n = 1,\dots,N$ there exists $\widetilde{\varphi}_n \in C^1([0,T])$ such that

$$\int_0^T (\widetilde{f}_n(t) - \widetilde{\varphi}_n(t))^2 dt \le \frac{\varepsilon^2}{2^{n+2}}, \quad n = 1,\dots,N.$$

This immediately implies that the function $\varphi = \sum_{n=1}^N \widetilde{\varphi}_n e_n$ is absolutely continuous, $\dot{\varphi} = \sum_{n=1}^N \widetilde{\varphi}_n' e_n \in L_2(U)$ and

$$\|f_N - \varphi\|_{L_2(U)} < \frac{\varepsilon}{2}.$$

Thus, $\|f - \varphi\|_{L_2(U)} < \varepsilon$, that finishes the proof of the lemma. □

LEMMA A.3. *If for $\varphi \in C_{\mathrm{id}}([0,T], L_2(\rho))$ and $k \in L_2([0,T], L_2(\lambda))$*

$$(h(T),\varphi(T))_{L_2(\lambda)} - (h(0),\mathrm{id})_{L_2(\lambda)} - \int_0^T (\dot{h}(s),\varphi(s))_{L_2(\lambda)} ds$$
$$= \int_0^T (h(s),k(s))_{L_2(\lambda)} ds, \quad \forall h \in H,$$

then φ belongs to $C_{\mathrm{id}}([0,T], L_2(\lambda))$ and is absolutely continuous with $\dot{\varphi} = k$.

PROOF. Set
$$\psi(t) = \mathrm{id} + \int_0^t k(s)ds, \quad t \in [0,T].$$

Then by Theorem 23.23 [49], $\psi \in C_{\mathrm{id}}([0,T], L_2(\lambda))$ and

$$\int_0^T (h(s),k(s))_{L_2(\lambda)} ds = (h(T),\psi(T))_{L_2(\lambda)} - (h(0),\mathrm{id})_{L_2(\lambda)}$$
$$- \int_0^T (\dot{h}(s),\psi(s))_{L_2(\lambda)} ds$$

for all $h \in H$. Hence, for all $h \in H$

$$(A.2)\quad \begin{aligned} &(h(T), \varphi(T))_{L_2(\lambda)} - \int_0^T (\dot{h}(s), \varphi(s))_{L_2(\lambda)} ds \\ &= (h(T), \psi(T))_{L_2(\lambda)} - \int_0^T (\dot{h}(s), \psi(s))_{L_2(\lambda)} ds. \end{aligned}$$

Taking $h(t) = h_0$, $t \in [0,T]$, where $h_0 \in L_2(\rho^{-1})$, we obtain

$$(h_0, \varphi(T))_{L_2(\lambda)} = (h_0, \psi(T))_{L_2(\lambda)}.$$

So, since $L_2(\rho^{-1}) \subset L_2(\lambda)$ densely, $\varphi(T) = \psi(T)$. Thus (A.2) can be rewritten as follows

$$\int_0^T (\dot{h}(s), \varphi(s))_{L_2(\lambda)} ds = \int_0^T (\dot{h}(s), \psi(s))_{L_2(\lambda)} ds.$$

This immediately implies $\varphi = \psi$ in $L_2([0,T], L_2(\lambda))$. Since ψ is continuous in $L_2(\lambda)$ it is also continuous in $L_2(\rho)$. Thus by continuity of φ, $\varphi(t) = \psi(t)$ in $L_2(\rho)$ for all $t \in [0,T]$. Consequently, $\varphi(t) = \psi(t)$ in $L_2(\lambda)$ for all $t \in [0,T]$. This finishes the proof of the lemma. □

APPENDIX B: CONVERGENCE OF THE FLOW OF PARTICLES WITH DRIFT

In this section we prove that the process $\{z^\varepsilon\}_{\varepsilon \in (0,1]}$ satisfying $(D^\varepsilon 1) - (D^\varepsilon 4)$ with $h = \dot{\varphi}$ tends to φ. Note that z^ε is a weak martingale solution to the equation

$$dz^\varepsilon(t) = \mathrm{pr}_{z^\varepsilon} \dot{\varphi}(t) dt + \sqrt{\varepsilon} \mathrm{pr}_{z^\varepsilon} dW_t.$$

If we show that z^ε converges to a process z taking values from $L_2^{\uparrow\uparrow}(\rho)$, then by Lemma 4.10, z should be a solution of the equation

$$dz(t) = \dot{\varphi}(t) dt,$$

It gives $z = \varphi$.

Thus, we prove first that the family $\{z^\varepsilon\}_{\varepsilon \in (0,1]}$ is tight. Then we show that any limit point z of $\{z^\varepsilon\}_{\varepsilon \in (0,1]}$ is $L_2^{\uparrow\uparrow}(\rho)$-valued process. As we noted, it immediately gives $z = \varphi$. Since $\{z^\varepsilon\}_{\varepsilon \in (0,1]}$ has only one nonrandom limit point, we obtain that $\{z^\varepsilon\}_{\varepsilon \in (0,1]}$ tends to φ in probability (not only in distribution).

PROPOSITION B.1. *Let $\varphi \in \mathscr{R}$ and a family of random elements $\{z^\varepsilon\}_{\varepsilon \in (0,1]}$ satisfies properties $(D^\varepsilon 1) - (D^\varepsilon 4)$ with $h = \dot{\varphi}$, then z^ε tends to φ in the space $C([0,T], L_2(\rho))$ in probability.*

To prove the proposition, we first establish tightness of $\{z^{\varepsilon}\}$ in $C([0,T],L_2(\rho))$, using the boundedness of $\dot{\varphi}$ and the same argument as in the proof of exponential tightness of $\{y^{\varepsilon}\}$. Next, testing the convergent subsequence $\{z^{\varepsilon'}\}$ by functions l from $C([0,1]\times[0,T],\mathbb{R})$ and using integration by parts we will obtain

$$\int_0^T\int_0^1 l(u,t)(z^{\varepsilon'}(u,t)-u)dtdu=\int_0^T\int_0^1 L(u,t)(\mathrm{pr}_{z^{\varepsilon'}(t)}\dot{\varphi}(t))(u)dtdu \\ +\int_0^T\int_0^1 L(u,t)d\eta^{\varepsilon'}(u,t)du,$$

where $L(u,t)=\int_t^T l(u,s)ds$ and $z^{\varepsilon'}\to z$. If $z(t)$ belongs to $L_2^{\uparrow\uparrow}(\rho)$ for all $t\in[0,T]$, then passing to the limit and using Lemma 4.10 we obtain

$$\int_0^T\int_0^1 l(u,t)(z(u,t)-u)dtdu=\int_0^T\int_0^1 L(u,t)\dot{\varphi}(u,t)dtdu,$$

which implies $z=\dot{\varphi}$.

The fact that $z(t)\in L_2^{\uparrow\uparrow}(\rho)$ will follow from the following lemma.

LEMMA B.2. *Let φ and $\{z^{\varepsilon}\}$ be such as in Proposition B.1. Then for each $u<v$ there exists $\delta>0$ such that*

$$\lim_{\varepsilon\to 0}\mathbb{P}\{z^{\varepsilon}(v,t)-z^{\varepsilon}(u,t)\le\delta\}=0.$$

Let $\mathfrak{S}(u,v,t)$ be a finite set of intervals contained in $[u,v]$ for all $t\in(0,T]$ such that

1) if $\pi_1,\pi_2\in\mathfrak{S}(u,v,t)$ and $\pi_1\neq\pi_2$, then $\pi_1\cap\pi_2=\emptyset$;
2) $\bigcup\mathfrak{S}(u,v,t)=[u,v]$;
3) for all $s<t$ and $\pi_1\in\mathfrak{S}(u,v,s)$ there exists $\pi_2\in\mathfrak{S}(u,v,t)$ that contains π_1;
4) there exists decreasing sequence $\{t_n\}_{n\ge1}$ on $(0,T]$ that tends to 0 and
$$\mathfrak{S}(u,v,t)=\mathfrak{S}(u,v,t_n),\quad t\in[t_n,t_{n-1}),\ \ n\in\mathbb{N},\ \ t_0=T;$$
5) for each monotone sequence $\pi(t)\in\mathfrak{S}(u,v,t)$, $t>0$, $\bigcap_{t>0}\pi(t)$ is a one-point set.

LEMMA B.3. *Let $\varphi\in\mathscr{R}$ and $[\widetilde{u},\widetilde{v}]\subset(0,1)$. Then there exists $\gamma>0$ such that for each interval $(u,v)\supset[\widetilde{u},\widetilde{v}]$ there exist $u_0\in(u,\widetilde{u})$ and $v_0\in(\widetilde{v},v)$:*

$$\inf_{t\in[0,T]}\left[v_0-u_0+\int_0^t(\mathrm{pr}_{\mathfrak{S}(s)}\dot{\varphi}(s))(v_0)ds-\int_0^t(\mathrm{pr}_{\mathfrak{S}(s)}\dot{\varphi}(s))(u_0)ds\right]=\delta>0,$$

for all $\mathfrak{S}(t)=\mathfrak{S}(0\vee(u-\gamma),(v+\gamma)\wedge 1,t)$, $t\in(0,T]$, such that u_0 and v_0 belong to separate intervals from $\mathfrak{S}(T)$, and $\mathrm{pr}_{\mathfrak{S}(t)}$ denotes the projection in $L_2(\lambda)$ onto the space of $\sigma(\mathfrak{S}(t))$-measurable functions.

PROOF. Let $u \in [0,1]$ and $\mathfrak{S}(t) = \mathfrak{S}(0,1,t)$, $t \in [0,T]$. Then we can choose a sequence of intervals $\{\pi_n\}_{n\geq 1}$ and a decreasing sequence $\{s_n\}_{n\geq 1}$ from $(0,T]$ converging to 0 such that $\pi_{n+1} \subseteq \pi_n \subseteq [0,1]$, $\{u\} = \bigcap_{n=1}^{\infty} \pi_n$ and

$$\begin{aligned} u + \int_0^t (\mathrm{pr}_{\mathfrak{S}(s)} \dot{\varphi}(s))(u) ds &= u + \sum_{n=1}^{\infty} \int_{s_n \wedge t}^{s_{n-1}\wedge t} \left(\frac{1}{|\pi_n|} \int_{\pi_n} \dot{\varphi}(q,r) dq \right) dr \\ &= u + \sum_{n=1}^{\infty} \frac{1}{|\pi_n|} \int_{\pi_n} (\varphi(q, s_{n-1} \wedge t) - \varphi(q, s_n \wedge t) dq \\ &= \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q,t) dq + \sum_{n=k}^{\infty} \left[\frac{1}{|\pi_{n+1}|} \int_{\pi_{n+1}} \varphi(q,s_n) dq \right. \\ &\quad \left. - \frac{1}{|\pi_n|} \int_{\pi_n} \varphi(q,s_n) dq \right], \end{aligned}$$

where $t \in [s_k, s_{k-1})$.

We estimate the n-th term of the sum. For convenience of calculations, let $[a,b] \subseteq [c,d]$ and $f : [0,1] \to \mathbb{R}$ be non-decreasing absolutely continuous function with bounded derivative. So,

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{d-c} \int_c^d f(x)dx &\leq \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{b-c} \int_c^b f(x)dx \\ &= \frac{a-c}{(b-a)(b-c)} \int_a^b f(x)dx - \frac{1}{b-c} \int_c^a f(x)dx \\ &\leq \frac{a-c}{b-c} f(b) - \frac{a-c}{b-c} f(c) = \frac{a-c}{b-c} (f(b) - f(c)) \\ &\leq \sup_{x\in[0,1]} \dot{f}(x)(a-c). \end{aligned}$$

Taking $c = a_n$, $d = b_n$, $a = a_{n+1}$, $b = b_{n+1}$ and $f = \varphi(\cdot, s_n)$, where $a_n < b_n$ are the ends of π_n, we get

$$\begin{aligned} u + \int_0^t (\mathrm{pr}_{\mathfrak{S}(s)} \dot{\varphi}(s))(u) ds &\leq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q,t) dq + \sum_{n=k}^{\infty} \widetilde{C}(a_{n+1} - a_n) \\ &\leq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q,t) dq + \widetilde{C}(u - a_k), \end{aligned}$$

where $\widetilde{C} = \sup_{(u,t)\in[0,1]\times[0,T]} \frac{\partial\varphi}{\partial u}(u,t)$.

Similarly, we can obtain

$$u + \int_0^t (\mathrm{pr}_{\mathfrak{S}(s)} \dot{\varphi}(s))(u) ds \geq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q,t) dq - \widetilde{C}(b_k - u).$$

Next, let $\widetilde{u}$ and $\widetilde{v}$ is from the statement of the lemma. Since φ is continuous on $[0,1]\times[0,T]$ and increasing by the first argument, the function

$$G(a,b,t)=\frac{1}{\widetilde{v}-b}\int_b^{\widetilde{v}}\varphi(q,t)dq-\frac{1}{a-\widetilde{u}}\int_{\widetilde{u}}^a\varphi(q,t)dq$$

is positive and continuous on $E=\{(a,b,t):\widetilde{u}\le a\le b\le\widetilde{v},\ t\in[0,T]\}$. Hence

$$\delta_1=\inf_E G>0.$$

Take $\gamma=\frac{\delta_1}{8\widetilde{C}}$ and for $u,v\in[0,1]$ satisfying $(u,v)\supset[\widetilde{u},\widetilde{v}]$ set

$$u_0=(u+\gamma)\wedge\widetilde{u},$$
$$v_0=(v-\gamma)\vee\widetilde{v}.$$

Let $\mathfrak{S}(t)=\mathfrak{S}(0\vee(u-\gamma),(v+\gamma)\wedge 1,t)$, $t\in(0,T]$, such that u_0 and v_0 belong to separate intervals from $\mathfrak{S}(T)$ and $t\in(0,T]$ be fixed. For π_1,π_2 belonging to $\mathfrak{S}(t)$ and containing u_0, v_0 respectively, we obtain

$$\begin{aligned}v_0-u_0&+\int_0^t(\mathrm{pr}_{\mathfrak{S}(s)}\dot{\varphi}(s))(v_0)ds-\int_0^t(\mathrm{pr}_{\mathfrak{S}(s)}\dot{\varphi}(s))(u_0)ds\\&\ge\frac{1}{|\pi_2|}\int_{\pi_2}\varphi(q,t)dq-\frac{1}{|\pi_1|}\int_{\pi_1}\varphi(q,t)dq\\&-\widetilde{C}(d-v_0)-\widetilde{C}(u_0-a)\\&\ge G(b\vee\widetilde{u},c\wedge\widetilde{v},t)-\widetilde{C}(d-v_0)-\widetilde{C}(u_0-a),\end{aligned}$$

where $c<d$ and $a<b$ are the ends of π_1 and π_2 respectively and $b\le c$ because u_0 and v_0 belong to separate intervals from $\mathfrak{S}(T)$. Since $u-\gamma\le a<u_0\le u+\gamma$ and $v-\gamma\le v_0<b\le v+\gamma$, we have

$$\begin{aligned}v_0-u_0&+\int_0^t(\mathrm{pr}_{\mathfrak{S}(s)}\dot{\varphi}(s))(v_0)ds-\int_0^t(\mathrm{pr}_{\mathfrak{S}(s)}\dot{\varphi}(s))(u_0)ds\\&\ge\delta_1-\widetilde{C}(v+\gamma-v+\gamma)-\widetilde{C}(u+\gamma-u+\gamma)\\&=\delta_1-4\widetilde{C}\gamma=\frac{\delta_1}{2}>0.\end{aligned}$$

It finishes the proof of the lemma. □

PROOF OF LEMMA B.2. Let $u<v$ be a fixed points from $(0,1)$ and δ, u_0, v_0, γ be defined in Lemma B.3 for some $[\widetilde{u},\widetilde{v}]\subset(u,v)$. Suppose that

$$\varliminf_{\varepsilon\to0}\mathbb{P}\left\{z^{\varepsilon}(v,t)-z^{\varepsilon}(u,t)\le\frac{\delta}{2}\right\}>0.$$

Set

$$B_1^\varepsilon = \{z^\varepsilon(u,t) = z^\varepsilon((u-\gamma)\vee 0,t)\},$$
$$B_2^\varepsilon = \{z^\varepsilon(v,t) = z^\varepsilon((v+\gamma)\wedge 1,t)\},$$
$$A^\varepsilon = \left\{z^\varepsilon(v,t) - z^\varepsilon(u,t) \le \frac{\delta}{2}\right\}.$$

Since the diffusion rate of $z^\varepsilon(u,\cdot)$ grows to infinity as the time tends to 0, it is convenient to work with the mean of z^ε because we can control the growing of diffusion rate in this case. So, denote

$$\xi_u^\varepsilon(t) = \frac{1}{u_0-u}\int_u^{u_0} z^\varepsilon(q,t)dq,$$
$$\xi_v^\varepsilon(t) = \frac{1}{v-v_0}\int_{v_0}^{u} z^\varepsilon(q,t)dq.$$

It is easy to see that

$$\widetilde{A}^\varepsilon = \left\{\xi_v^\varepsilon(t) - \xi_u^\varepsilon(t) \le \frac{\delta}{2}\right\} \supseteq A^\varepsilon.$$

Next, using the processes ξ_u^ε and ξ_v^ε, we want to show that

$$\lim_{\varepsilon\to 0}\mathbb{P}\{A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon)^c\} = 0. \tag{B.1}$$

Note that ξ_u^ε and ξ_v^ε are diffusion processes, namely

$$\xi_u^\varepsilon(t) = \frac{u_0+u}{2} + \int_0^t a_u^\varepsilon(s)ds + \chi_u^\varepsilon(t),$$
$$\xi_v^\varepsilon(t) = \frac{v_0+v}{2} + \int_0^t a_v^\varepsilon(s)ds + \chi_v^\varepsilon(t),$$

where

$$a_u^\varepsilon(t) = \frac{1}{u_0-u}\int_u^{u_0} (\mathrm{pr}_{z^\varepsilon(t)}\dot{\varphi}(t))(q)dq,$$
$$a_v^\varepsilon(t) = \frac{1}{v-v_0}\int_{v_0}^{v} (\mathrm{pr}_{z^\varepsilon(t)}\dot{\varphi}(t))(q)dq,$$
$$\chi_u^\varepsilon(t) = \frac{1}{u_0-u}\int_0^t\int_u^{u_0} d\eta^\varepsilon(q,s)dq,$$
$$\chi_v^\varepsilon(t) = \frac{1}{v-v_0}\int_0^t\int_{v_0}^{v} d\eta^\varepsilon(q,s)dq.$$

By choosing of $u_0,\ v_0$ and δ, we have

$$L^\varepsilon(t,\omega) = \frac{v_0+v}{2} - \frac{u_0+u}{2} \\ + \int_0^t a_v^\varepsilon(s,\omega)ds - \int_0^t a_u^\varepsilon(s,\omega)ds \geq \delta, \quad t\in[0,T], \ \ \omega\in(B_1^\varepsilon\cup B_2^\varepsilon)^c.$$

Denote the difference $\xi_v^\varepsilon - \xi_u^\varepsilon$ by ξ^ε. Note that the quadratic variation of the martingale part χ^ε of ξ^ε satisfies

$$[\chi^\varepsilon]_t \leq \varepsilon C t,$$

where $C = \frac{1}{u_0-u} + \frac{1}{v-v_0}$. So, denoting

$$\sigma^\varepsilon = \inf\left\{t:\ \xi^\varepsilon(t) = \frac{\delta}{2}\right\},$$

we get

$$\mathbb{P}\{A^\varepsilon\cap(B_1^\varepsilon\cup B_2^\varepsilon)^c\} \leq \mathbb{P}\{\{\sigma^\varepsilon\leq t\}\cap(B_1^\varepsilon\cup B_2^\varepsilon)^c\},$$

which implies (B.1). Indeed, by Theorem 2.7.2 [22], there exists a standard Wiener process $w^\varepsilon(t),\ t\geq 0$, such that

$$\chi^\varepsilon(t) = w^\varepsilon([\chi^\varepsilon]_t).$$

Define $\tau^\varepsilon = \inf\left\{t: w^\varepsilon(\varepsilon C t) = -\frac{\delta}{2}\right\}$. Since

$$\xi^\varepsilon(t) = \delta + (L^\varepsilon(t)-\delta) + \chi^\varepsilon(t)$$

and the term $L^\varepsilon - \delta$ is non-negative on $(B_1^\varepsilon\cup B_2^\varepsilon)^c$, the process $\delta + w^\varepsilon(\varepsilon C\cdot)$ hits at the point $\frac{\delta}{2}$ sooner than ξ^ε. Consequently, $\{\sigma^\varepsilon\leq t\}\cap(B_1^\varepsilon\cup B_2^\varepsilon)^c \subseteq \{\tau^\varepsilon\leq t\}\cap(B_1^\varepsilon\cup B_2^\varepsilon)^c$. This yields (B.1).

Next, the relation $\mathbb{P}\{A^\varepsilon\cap(B_1^\varepsilon\cup B_2^\varepsilon)\} = \mathbb{P}\{A^\varepsilon\} - \mathbb{P}\{A^\varepsilon\cap(B_1^\varepsilon\cup B_2^\varepsilon)^c\}$, (B.1) and the assumption $\underline{\lim}_{\varepsilon\to 0}\mathbb{P}\{A^\varepsilon\}>0$ imply

$$\underline{\lim}_{\varepsilon\to 0}\mathbb{P}\{A^\varepsilon\cap(B_1^\varepsilon\cup B_2^\varepsilon)\}>0.$$

Thus, we obtain

$$\underline{\lim}_{\varepsilon\to 0}\mathbb{P}\{A^\varepsilon\cap B_1^\varepsilon\}>0 \ \text{ or } \ \underline{\lim}_{\varepsilon\to 0}\mathbb{P}\{A^\varepsilon\cap B_2^\varepsilon\}>0.$$

It means that we can extend the interval $[u,v]$ to $[(u-\gamma)\vee 0, v]$ or $[u,(v+\gamma)\wedge 1]$, i.e.

$$\underline{\lim}_{\varepsilon\to 0}\mathbb{P}\left\{z^\varepsilon(v,t) - z^\varepsilon((u-\gamma)\vee 0,t) \leq \frac{\delta}{2}\right\}>0$$

or

$$\varliminf_{\varepsilon\to 0}\mathbb{P}\left\{z^{\varepsilon}((v+\gamma)\wedge 1,t)-z^{\varepsilon}(u,t)\le\frac{\delta}{2}\right\}>0.$$

Noting that γ only depends on $[\widetilde{u},\widetilde{v}]$ and applying the same argument for new start points of the particles in finitely many steps, we obtain

$$\varliminf_{\varepsilon\to 0}\mathbb{P}\left\{z^{\varepsilon}(v_1,t)-z^{\varepsilon}(u_1,t)\le\frac{\delta}{2}\right\}>0,$$

where $(u_1,v_1)\supset[\widetilde{u},\widetilde{v}]$ and $u_1=0$ or $v_1=1$. Next, applying the same argument for new start points of the particles, but replacing $B_1^{\varepsilon}\cup B_2^{\varepsilon}$ by B_1^{ε}, if $v_1=1$ or B_2^{ε}, if $u_1=0$, in finitely many steps we get

$$\varliminf_{\varepsilon\to 0}\mathbb{P}\left\{z^{\varepsilon}(1,t)-z^{\varepsilon}(0,t)\le\frac{\delta}{2}\right\}>0.$$

But it is not possible because the same argument (without B_1^{ε} and B_2^{ε}) gives

$$\lim_{\varepsilon\to 0}\mathbb{P}\left\{z^{\varepsilon}(1,t)-z^{\varepsilon}(0,t)\le\frac{\delta}{2}\right\}=0.$$

The lemma is proved. $\square$

PROOF OF PROPOSITION B.1. Using Jakubowski's tightness criterion (see Theorem 3.1 [24]) and boundedness of $\dot{\varphi}$, as in the proof of exponential tightness of $\{y^{\varepsilon}\}$ (see Proposition 4.5), we can prove that $\{z^{\varepsilon}\}_{\varepsilon\in(0,1]}$ is tight in $C([0,T],L_2(\rho))$.

Let $\{z^{\varepsilon'}\}$ be a convergent subsequence and z is its limit. By Skorokhod's theorem (see Theorem 3.1.8 [19]), we can define a probability space and a sequence of random elements $\{\widetilde{z}^{\varepsilon'}\}$, $\widetilde{z}$ on this space such that $\mathrm{Law}(z^{\varepsilon'})=\mathrm{Law}(\widetilde{z}^{\varepsilon'})$, $\mathrm{Law}(z)=\mathrm{Law}(\widetilde{z})$ and $\widetilde{z}^{\varepsilon'}\to\widetilde{z}$ in $C([0,T],L_2(\rho))$ a.s. If we show that $\widetilde{z}=\varphi$, we finish the proof because this implies that $\widetilde{z}^{\varepsilon'}\to\varphi$ in $C([0,T],L_2(\rho))$ in probability and since φ is non-random, $z^{\varepsilon'}\to\varphi$ in probability. Thus, it will easily yield that $z^{\varepsilon}\to\varphi$ in probability.

So, for convenience of notation we will assume that $z^{\varepsilon}\to z$ a.s., instead $\widetilde{z}^{\varepsilon'}\to\widetilde{z}$. First we check that $z(t)\in L_2^{\uparrow\uparrow}(\rho)$ for all $t\in[0,T]$. Let t is fixed. One can show that

$$z^{\varepsilon}(t)\to z(t)\quad\text{in measure }\mathbb{P}\otimes\rho.$$

By Lemma 4.2 [25], there exists subsequence $\{\varepsilon'\}$ such that

$$z^{\varepsilon'}(t)\to z(t)\quad\mathbb{P}\otimes\rho-\text{a.e.}$$

Set $A=\{(\omega,u):\ z^{\varepsilon'}(u,t,\omega)\to z(u,t,\omega)\}$. Since $\mathbb{P}\otimes\rho(A^c)=0$, it is easy to see that there exists the set $U\subseteq[0,1]$ such that $\rho(U^c)=0$ and $\mathbb{P}(A_u)=1$, for all $u\in U$, where $A_u=\{\omega:\ (\omega,u)\in A\}$. Note, it implies that for each $u\in U$

$$z^{\varepsilon'}(u,t)\to z(u,t)\quad \text{a.s.}$$

Let U_{count} is a countable subset of U which is dense in $[0,1]$. From Lemma B.2 it follows that

$$z(u,t)<z(v,t)\quad \text{a.s.},$$

for all $u,v\in U_{count}$, $u<v$.

Denote

$$\Omega'=\bigcap_{u<v,\ u,v\in U_{count}}\{z(u,t)<z(v,t)\}.$$

Since U_{count} is countable, $\mathbb{P}(\Omega')=1$. Next, define

$$\widetilde{z}(u,t,\omega)=\inf_{u\le v,\ v\in U_{count}} z(v,t,\omega),\quad u\in[0,1],\ \omega\in\Omega'.$$

Then for all $\omega\in\Omega'$, $\widetilde{z}(\cdot,t,\omega)\in D^{\uparrow\uparrow}$. Let us show that $\rho\{u:\ \widetilde{z}(u,t)\neq z(u,t)\}=0$ a.s.

Denote

$$\widetilde{\Omega}=\left(\bigcap_{u\in U_{count}}A_u\right)\cap\Omega'\cap\{z^{\varepsilon'}(t)\to z(t)\text{ in }L_2(\rho)\}.$$

Then

$$z^{\varepsilon}(u,t,\omega)\to z(u,t,\omega)=\widetilde{z}(u,t,\omega)\tag{B.2}$$

for all $u\in U_{count}$ and $\omega\in\widetilde{\Omega}$. Fix $\omega\in\widetilde{\Omega}$. Since $\widetilde{z}(\cdot,t,\omega)$ is nondecreasing, it has a countable set $D_{\widetilde{z}(\cdot,t,\omega)}$ of discontinuous points. The countability implies that $\rho(D_{\widetilde{z}(\cdot,t,\omega)})=0$. Take $u\in D^c_{\widetilde{z}(\cdot,t,\omega)}$, then from monotonicity of $\widetilde{z}(\cdot,t,\omega)$ and $z^{\varepsilon'}(\cdot,t,\omega)$, density of U_{count} and (B.2) we can obtain

$$z^{\varepsilon}(u,t,\omega)\to\widetilde{z}(u,t,\omega).$$

Thus,

$$z^{\varepsilon}(\cdot,t,\omega)\to\widetilde{z}(\cdot,t,\omega)\quad \rho-\text{a.e.}$$

On the other hand,

$$z^{\varepsilon}(\cdot,t,\omega)\to z(\cdot,t,\omega)\quad \text{in }L_2(\rho).$$

Consequently, $z(\cdot,t,\omega)=\widetilde{z}(\cdot,t,\omega)$ ρ-a.e. for all $\omega\in\widetilde{\Omega}$. So, it means that $z(t)\in L_2^{\uparrow\uparrow}(\rho)$ a.s., for all $t\in[0,T]$.

Now we can prove that $z = \varphi$. Take $l \in C([0,1] \times [0,T], \mathbb{R})$. By the dominated convergence theorem, $\int_0^T \int_0^1 l(u,t)(z^\varepsilon(u,t) - u)dtdu$ converges to $\int_0^T \int_0^1 l(u,t)(z(u,t) - u)dtdu$ a.s. Integrating by parts, we get

$$\int_0^T \int_0^1 l(u,t)(z^\varepsilon(u,t) - u)dtdu = \int_0^T \int_0^1 L(u,t)(\mathrm{pr}_{z^\varepsilon(t)} \dot{\varphi}(t))(u)dtdu + \int_0^T \int_0^1 L(u,t)d\eta^\varepsilon(u,t)du,$$

where $L(u,t) = \int_t^T l(u,s)ds$. The first term in the right hand side of the previous relation converges to $\int_0^T \int_0^1 L(u,t)\dot{\varphi}(u,t)dtdu$, by Lemma 4.10. The second term is the stochastic integral so we can estimate the expectation of its second moment

$$\mathbb{E}\left(\int_0^T \int_0^1 L(u,t)d\eta^\varepsilon(u,t)du\right)^2 \leq \varepsilon\mathbb{E}\int_0^T \int_0^1 (\mathrm{pr}_{z^\varepsilon(t)} L(t))^2(u)dtdu \leq \varepsilon\mathbb{E}\int_0^T \int_0^1 L^2(u,t)dtdu \to 0, \quad \varepsilon \to 0.$$

Consequently, we obtain

$$\int_0^T \int_0^1 l(u,t)(z(u,t) - u)dtdu = \int_0^T \int_0^1 L(u,t)\dot{\varphi}(u,t)dtdu,$$

which easily implies $z = \varphi$. The proposition is proved. □

REFERENCES

[1] ADAMS, S., DIRR, N., PELETIER, M. A. and ZIMMER, J. (2011). From a large-deviations principle to the Wasserstein gradient flow: a new micro-macro passage. *Comm. Math. Phys.* **307** 791–815. MR2842966

[2] ALONSO, A. and BRAMBILA-PAZ, F. (1998). L^p-continuity of conditional expectations. *J. Math. Anal. Appl.* **221** 161–176. MR1619139

[3] ANDRES, S. and VON RENESSE, M.-K. (2010). Particle approximation of the Wasserstein diffusion. *J. Funct. Anal.* **258** 3879–3905. MR2606878

[4] ARRATIA, R. A. (1979). *Coalescing Brownian motion on the line*. ProQuest LLC, Ann Arbor, MI Thesis (Ph.D.)–The University of Wisconsin - Madison. MR2630231

[5] BENAMOU, J.-D. and BRENIER, Y. (2000). A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.* **84** 375–393. MR1738163

[6] BRENIER, Y., GANGBO, W., SAVARÉ, G. and WESTDICKENBERG, M. (2013). Sticky particle dynamics with interactions. *J. Math. Pures Appl. (9)* **99** 577–617. MR3039208

[7] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic equations in infinite dimensions*, second ed. *Encyclopedia of Mathematics and its Applications* **152**. Cambridge University Press, Cambridge. MR3236753

[8] DAWSON, D. A. (1993). Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI—1991*. *Lecture Notes in Math.* **1541** 1–260. Springer, Berlin. MR1242575

[9] DEAN, D. S. (1996). Langevin equation for the density of a system of interacting Langevin processes. *J. Phys. A* **29** L613–L617. MR1446882

[10] DEMBO, A. and ZEITOUNI, O. (2010). *Large deviations techniques and applications*. *Stochastic Modelling and Applied Probability* **38**. Springer-Verlag, Berlin Corrected reprint of the second (1998) edition. MR2571413
[11] DJEHICHE, B. and SCHIED, A. (1998). Large deviations for hierarchical systems of interacting jump processes. *J. Theoret. Probab.* **11** 1–24. MR1607396
[12] DONATI-MARTIN, C. (2008). Large deviations for Wishart processes. *Probab. Math. Statist.* **28** 325–343. MR2548976
[13] DONATI-MARTIN, C., ROUAULT, A., YOR, M. and ZANI, M. (2004). Large deviations for squares of Bessel and Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields* **129** 261–289. MR2063378
[14] DOROGOVTSEV, A. A. (2004). One Brownian stochastic flow. *Theory Stoch. Process.* **10** 21–25. MR2329772
[15] DOROGOVTSEV, A. A. (2007). *Measure-valued processes and stochastic flows* **66**. Institute of Mathematics of NAS of Ukraine, Kiev (in Russian). MR2375817
[16] DOROGOVTSEV, A. A. and NISHCHENKO, I. I. (2014). An analysis of stochastic flows. *Commun. Stoch. Anal.* **8** 331–342. MR3310624
[17] DOROGOVTSEV, A. A. and OSTAPENKO, O. V. (2010). Large deviations for flows of interacting Brownian motions. *Stoch. Dyn.* **10** 315–339. MR2671379
[18] DRIVER, B. (2003). *Analysis tools with applications*. E-Book. Springer, Berlin.
[19] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov processes*. *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. John Wiley & Sons, Inc., New York Characterization and convergence. MR838085
[20] FONTES, L. R. G., ISOPI, M., NEWMAN, C. M. and RAVISHANKAR, K. (2004). The Brownian web: characterization and convergence. *Ann. Probab.* **32** 2857–2883. MR2094432
[21] HIDA, T. (1980). *Brownian motion*. *Applications of Mathematics* **11**. Springer-Verlag, New York-Berlin Translated from the Japanese by the author and T. P. Speed. MR562914
[22] IKEDA, N. and WATANABE, S. (1981). *Stochastic differential equations and diffusion processes*. *North-Holland Mathematical Library* **24**. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo. MR637061
[23] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit theorems for stochastic processes*, second ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer-Verlag, Berlin. MR1943877
[24] JAKUBOWSKI, A. (1986). On the Skorokhod topology. *Ann. Inst. H. Poincaré Probab. Statist.* **22** 263–285. MR871083
[25] KALLENBERG, O. (2002). *Foundations of modern probability*, second ed. *Probability and its Applications (New York)*. Springer-Verlag, New York. MR1876169
[26] KAWASAKI, K. (1994). Stochastic model of slow dynamics in supercooled liquids and dense colloidal suspensions. *Physica A: Statistical Mechanics and its Applications* **208** 35 - 64.
[27] KONAROVSKIĬ, V. V. (2010). On an infinite system of diffusing particles with coalescing. *Teor. Veroyatn. Primen.* **55** 157–167. MR2768524
[28] KONAROVSKYI, V. A system of coalescing heavy diffusion particles on the real line. *To appear in Ann. Probab.* arXiv:1408.0628v3.
[29] KONAROVSKYI, V. V. (2014). Large deviations principle for finite system of heavy diffusion particles. *Theory Stoch. Process.* **19** 37–45. MR3337132
[30] KONAROVSKYI, V. (2017). On asymptotic behavior of the modified Arratia flow. *Electron. J. Probab.* **22** Paper No. 19, 31. MR3622889
[31] LE JAN, Y. and RAIMOND, O. (2004). Flows, coalescence and noise. *Ann. Probab.* **32** 1247–1315. MR2060298
[32] LOTT, J. and VILLANI, C. (2009). Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)* **169** 903–991. MR2480619

[33] MALOVICHKO, T. V. (2009). Girsanov's theorem for stochastic flows with interaction. *Ukraïn. Mat. Zh.* **61** 365–383. MR2562218

[34] OTTO, F. (2001). The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations* **26** 101–174. MR1842429

[35] OTTO, F. and VILLANI, C. (2000). Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173** 361–400. MR1760620

[36] REVUZ, D. and YOR, M. (1999). *Continuous martingales and Brownian motion*, third ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **293**. Springer-Verlag, Berlin. MR1725357

[37] SCHIED, A. (1995). *Criteria for exponential tightness in path spaces*. Unpublished preprint.

[38] SCHIED, A. (1997). Moderate deviations and functional LIL for super-Brownian motion. *Stochastic Process. Appl.* **72** 11–25. MR1483609

[39] SHAMOV, A. (2011). Short-time asymptotics of one-dimensional Harris flows. *Commun. Stoch. Anal.* **5** 527–539. MR2840266

[40] STANNAT, W. (2013). Two remarks on the Wasserstein Dirichlet form. In *Seminar on Stochastic Analysis, Random Fields and Applications VII*. *Progr. Probab.* **67** 235–255. Birkhäuser/Springer, Basel. MR3380344

[41] STURM, K.-T. (2006). On the geometry of metric measure spaces. I. *Acta Math.* **196** 65–131. MR2237206

[42] STURM, K.-T. (2014). A monotone approximation to the Wasserstein diffusion. In *Singular phenomena and scaling in mathematical models* 25–48. Springer, Cham. MR3205035

[43] VAKHANIA, N. N., TARIELADZE, V. I. and CHOBANYAN, S. A. (1987). *Probability distributions on Banach spaces*. *Mathematics and its Applications (Soviet Series)* **14**. D. Reidel Publishing Co., Dordrecht Translated from the Russian and with a preface by Wojbor A. Woyczynski. MR1435288

[44] VARADHAN, S. R. S. (1967). On the behavior of the fundamental solution of the heat equation with variable coefficients. *Comm. Pure Appl. Math.* **20** 431–455. MR0208191

[45] VILLANI, C. (2009). *Optimal transport. Old and new*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **338**. Springer-Verlag, Berlin. MR2459454

[46] VON RENESSE, M.-K. (2012). An optimal transport view of Schrödinger's equation. *Canad. Math. Bull.* **55** 858–869. MR2994690

[47] VON RENESSE, M.-K. and STURM, K.-T. (2005). Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.* **58** 923–940. MR2142879

[48] VON RENESSE, M.-K. and STURM, K.-T. (2009). Entropic measure and Wasserstein diffusion. *Ann. Probab.* **37** 1114–1191. MR2537551

[49] ZEIDLER, E. (1990). *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York Linear monotone operators, Translated from the German by the author and Leo F. Boron. MR1033497

Max Planck Institut für Mathematik
in den Naturwissenschaften,
Inselstrae 22,
04103 Leipzig,
Germany
and
Department of Mathematics and
Informatics,
Yuriy Fedkovych Chernivtsi
National University,
Kotsubinsky Str. 2,
58012 Chernivtsi,
Ukraine
e-mail: konarovskiy@gmail.com

Universität Leipzig,
Fakultät für Mathematik
und Informatik,
Augustusplatz 10,
04109 Leipzig,
Germany
e-mail: renesse@math.uni-leipzig.de