A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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Recent Developments in SPDEs and BSDEs meet Harmonic and Functional Analysis

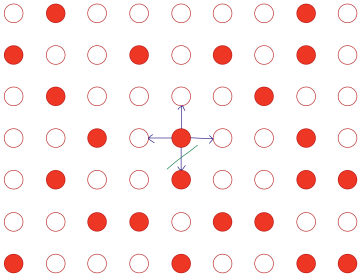
joint work with Benjamin Gess

Simple symmetric exclusion process

On the d-dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : \ k \in \mathbb{Z}_n^d := \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

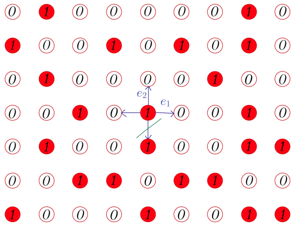
we consider a **Simple Symmetric Exclusion Process** (SSEP)



State space and generator

Particle configuration $\eta \in \{0,1\}^{\mathbb{T}_n^d}$:

$$\eta(x) = 0 \Leftrightarrow \text{side } x \text{ is empty}$$
$$\eta(x) = 1 \Leftrightarrow \text{side } x \text{ is occupied}$$



$$\eta^{x\leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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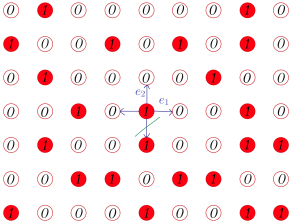

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$$\mathcal{G}_n^{EP} F(\eta) := \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x \leftrightarrow x+e_j}) - F(\eta) \right]$$ [Kipnis, Landim '99]

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

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Let η_t^n, $t \geq 0$, be a SSEP and $\rho_0 : \mathbb{T}^d \to [0, 1]$ be an initial profile.

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The process $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$ solves the discrete stochastic Heat equation

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Thus,

$$\tilde{\rho}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_t^n(x)\delta_x \to \rho_t^\infty,$$

where $\rho_t^\infty := P_t^{HE}\rho_0$ solves

$$d\rho_t^\infty = \frac{1}{2}\Delta\rho_t^\infty dt, \quad \rho_0^\infty = \rho_0.$$

Law of large numbers

Theorem **[see e.g. in Kipnis, Landim '99]**

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho_0(x))$ be independent. Then

$$\tilde{\eta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x, \quad t \geq 0$$

converges in probability to $\rho_t^\infty(x)$, $t \geq 0$, where $\rho_t^\infty := P_t^{HE} \rho_0$ solves

$$d\rho_t^\infty = \frac{1}{2} \Delta \rho_t^\infty dt, \quad \rho_0^\infty = \rho_0.$$

Convergence of generator

Note that $\langle \varphi, \tilde{\eta}^n_t \rangle$ solves the martingale problem

$$f\left(\langle \varphi, \tilde{\eta}^n_t \rangle\right) - \int_0^t \mathcal{G}^{EP}_n f\left(\langle \varphi, \tilde{\eta}^n_s \rangle\right) ds \quad \text{is a mart.,}$$

where

$$\begin{aligned}
\mathcal{G}^{EP}_n f\left(\langle \varphi, \tilde{\eta} \rangle\right) &:= \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[f\left(\left\langle \varphi, \tilde{\eta}^{x \leftrightarrow x+e_j} \right\rangle\right) - f\left(\langle \varphi, \tilde{\eta} \rangle\right)\right] \\
&= \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} f'\left(\langle \varphi, \tilde{\eta} \rangle\right) \underbrace{\left(\left\langle \varphi, \tilde{\eta}^{x \leftrightarrow x+e_j} \right\rangle - \langle \varphi, \tilde{\eta} \rangle\right)}_{\left[\varphi(x+e_j) - \varphi(x)\right]\left[\eta(x) - \eta(x+e_j)\right]} \\
&\qquad + \frac{n^2}{4} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} f''\left(\langle \varphi, \tilde{\eta} \rangle\right) \left(\left\langle \varphi, \tilde{\eta}^{x \leftrightarrow x+e_j} \right\rangle - \langle \varphi, \tilde{\eta} \rangle\right)^2 \\
&= \frac{1}{2} f'\left(\langle \varphi, \tilde{\eta} \rangle\right) \langle \Delta_n \varphi, \tilde{\eta} \rangle \\
&\qquad + \frac{1}{4n^d} f''\left(\langle \varphi, \tilde{\eta} \rangle\right) \sum_{j=1}^{d} \left\langle \left|\partial_{n,j} \varphi\right|^2, \widetilde{\tau_j \eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \right\rangle + \ldots,
\end{aligned}$$

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n(x) := n^{\frac{d}{2}} \left(\eta_t^n(x) - \rho_t^n(x)\right).$$

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The generator of

$$\tilde{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

can be expanded as follows

$$\mathcal{G}_n^{FF} f\left(\langle \varphi, \tilde{\zeta} \rangle\right) = \frac{1}{2} f'\left(\langle \varphi, \tilde{\zeta} \rangle\right) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f''\left(\langle \varphi, \tilde{\zeta} \rangle\right) \sum_{j=1}^{d} \left\langle |\partial_{n,j} \varphi|^2, \widetilde{\tau_j \eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \right\rangle$$

$$+ O\left(1/n^{\frac{d}{2}+1}\right)$$

$$\xrightarrow[n \to \infty]{}$$

$$\mathcal{G}^{SPDE} f\left(\langle \varphi, \tilde{\zeta} \rangle\right) = \frac{1}{2} f'\left(\langle \varphi, \tilde{\zeta} \rangle\right) \langle \Delta \varphi, \tilde{\zeta} \rangle + \frac{1}{4} f''\left(\langle \varphi, \tilde{\zeta} \rangle\right) \left\langle |\nabla \varphi|^2, \tilde{\rho} + \tilde{\rho} - 2\tilde{\rho}^2 \right\rangle$$

Central limit theorem

Theorem 2 **[Galves, Kipnis, Spohn; Ravishankar '90]**

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\tilde{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D\left([0, T], \mathcal{D}'\right)$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E}\left[\langle \zeta_0^\infty, \varphi \rangle^2 \right] = \langle \rho_0 (1 - \rho_0) \varphi, \varphi \rangle$$

Our goal

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- [Chassagneux, Szpruch, Tse '22]: Weak quantitative propagation of chaos (mean field limit)
- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho_0(x))$ and independent.

Then

$$\sup_{t\in[0,T]} \left| \mathbb{E} f\left(\langle \vec{\varphi}, \tilde{\zeta}_t^n \rangle\right) - \mathbb{E} f\left(\langle \vec{\varphi}, \zeta_t^\infty \rangle\right) \right| \le \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}_l^3} \|\vec{\varphi}\|_{\mathrm{C}^l}$$

for all $n \ge 1$, $f \in \mathrm{C}_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^l(\mathbb{T}^d)\right)^m$, where l is large enough.

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The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ is optimal:

$\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t taking values in the same state space and $X_0 = Y_0 = x$ using

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t P_s^X \left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(x) ds,$$
$$= \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(X_s)\right] ds,$$

[see e.g. Ethier, Kurtz '86]

Splitting of the problem

Recall

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We consider the Markov processes:

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We will compare:

- $X_t := (\tilde{\rho}^n_t, \tilde{\zeta}^n_t)$ an $Y_t := (\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ **[comparison of dynamics]** where the generalized OU process $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ started from $(\tilde{\rho}^n_0, \tilde{\zeta}^n_0)$;
- $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ and $(\rho^\infty_t, \zeta^\infty_t)$ **[comparison of initial conditions]** (both are defined by the same equation).

Generators

We start from the formal computation for cylindrical functions:

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$$\mathcal{G}_n^{FF} F(\tilde{\rho},\tilde{\zeta}) = \ldots + \frac{1}{4}\partial_2^2 f \left\langle \left|\partial_{n,j}\varphi\right|^2, \tau_j\tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta\tau_j\eta} \right\rangle + O\left(1/n^{\frac{d}{2}+1}\right),$$
$$\mathcal{G}^{OU} F(\tilde{\rho},\tilde{\zeta}) = \ldots + \frac{1}{4}\partial_2^2 f \left\langle \left|\partial_j\varphi\right|^2, 2\tilde{\rho} - 2\tilde{\rho}^2 \right\rangle.$$

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Thus

$$\left(\mathcal{G}_n^{FF}-\mathcal{G}^{OU}\right)F(\tilde\rho,\tilde\zeta) = \ldots + \frac{1}{4}\partial_2^2 f\left[\left\langle\left|\partial_{n,j}\varphi\right|^2, \tau_j\tilde\eta+\tilde\eta-2\widetilde{\eta\tau_j\eta}\right\rangle - \left\langle\left|\partial_j\varphi\right|^2, 2\tilde\rho-2\tilde\rho^2\right\rangle\right]$$
$$+ O\left(\frac{1}{n^{\frac{d}{2}+1}}\right).$$

The difficulties

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We need different lifting of the particle system to the Sobolev spaces.

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x)\delta_x$ and $\tilde{\zeta} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x)\delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
  – basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
  – eigenvectors for discrete and continuous diff. operators

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x)\delta_x$ and $\tilde{\zeta} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x)\delta_x$ by a smooth interpolation.

- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

$$\langle \rho_1, \rho_2 \rangle_n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_1(x)\rho_2(x)$$

- $L_2(\mathbb{T}^d)$ be the usual L_2-space of function on $\mathbb{T}^d$ with

$$\langle g_1, g_2 \rangle := \int_{\mathbb{T}^d} g_1(x) g_2(x) dx.$$

- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
  – basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
  – eigenvectors for discrete and continuous diff. operators

$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \ \text{ on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle \varsigma_k \ \text{ on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\mathrm{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \mathrm{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\mathrm{ex}_n f$ and $\mathrm{pr}_n g$

- $\mathrm{ex}_n\rho = \rho$ on $\mathbb{T}_n^d$ and $\mathrm{ex}_n\rho \in \mathrm{C}^\infty(\mathbb{T}^d)$
- $\mathrm{pr}_n\varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2\rangle_n = \langle \mathrm{ex}_n\rho_1, \mathrm{ex}_n\rho_2\rangle$ and $\langle \rho, \mathrm{pr}_n g\rangle_n = \langle \mathrm{ex}_n\rho, g\rangle$
- $\|\mathrm{pr}_n g - g\|_{H_J} \le \frac{1}{n}\|g\|_{H_{J+1}}$, $\|\mathrm{ex}_n\varphi - \varphi\|_{H_J} \le \frac{C}{n}\|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}, \dots$

Comparison of generators for smooth interpolations

We replace

$$\tilde{\rho} \rightsquigarrow \mathrm{ex}_n\rho =: \hat{\rho}, \qquad \tilde{\zeta} \rightsquigarrow \mathrm{ex}_n\zeta =: \hat{\zeta}$$

in particular

$$\langle \varphi, \mathrm{ex}_n\zeta \rangle = \langle \mathrm{pr}_n\varphi, \zeta \rangle_n = \langle \mathrm{pr}_n\varphi, \tilde{\zeta} \rangle$$

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Thus

$$\begin{aligned}\left(\mathcal{G}_n^{FF} - \mathcal{G}^{OU}\right) F(\hat{\rho}, \hat{\zeta}) &= \frac{1}{2} \partial_1 f[\ldots] \\ &+ \frac{1}{4} \partial_2^2 f \left[\left\langle \mathrm{ex}_n \left| \partial_{n,j} \mathrm{pr}_n \varphi \right|^2, \widehat{\tau_j \eta} + \hat{\eta} - 2 \mathrm{ex}_n (\eta \tau_j \eta) \right\rangle - \left\langle \left| \partial_j \varphi \right|^2, 2\hat{\rho} - 2\hat{\rho}^2 \right\rangle \right] \\ &+ O\left(\frac{1}{n^{\frac{d}{2}+1}} \right).\end{aligned}$$

Overcoming of problems

We deal with the most problematic term $\mathrm{ex}_n(\eta\tau_j\eta) - \hat{\rho}^2$ as follows

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- The term $\mathbb{E}\left\langle \frac{1}{n^d}\mathrm{ex}_n(\zeta_t^n\tau_j\zeta_t^n), \dots \right\rangle^2$ can be controlled via 4-point correlation function

$$\mathbb{E}\prod_{i=1}^{4}(\eta_t^n(x_i) - \rho_t^n(x_i)) \lesssim \frac{1}{n}$$

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All computations and estimates for $\left(\mathcal{G}_n^{FF} - \mathcal{G}^{OU}\right)F(\hat{\rho}, \hat{\zeta})$ can be easily transferred to the case $F \in \mathrm{C}^{1,3}(H_J \times H_{-I})$.

Differentiability of $P_t^{OU}F(\hat{\rho}, \hat{\zeta})$

A solution to

$$
\begin{aligned}
d\rho_t^\infty &= \frac{1}{2}\Delta\rho_t^\infty dt \\
d\zeta_t^\infty &= \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)
\end{aligned}
$$

exists for all $\rho_0^\infty \in L_2(\mathbb{T}^d; [0,1])$ and $\zeta_0^\infty \in H_{-l}$ for $l > \frac{d}{2} + 1$.

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For $F \in \mathrm{C}(H_J \times H_{-l})$ (e.g. $F = f(\langle \psi, \cdot \rangle, \langle \varphi, \cdot \rangle)$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E} F\left(\rho_t^\infty, \zeta_t^\infty\right)$

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Proposition [Gess, K. '24]

Let $I > \frac{d}{2} + 1$ and $F \in \mathrm{C}_b^{2,4}(H_{-I})$. Then $U_t(\rho_0^\infty, \zeta_0^\infty) = \mathbb{E} F(\zeta_t^\infty) \in \mathrm{C}_b^{1,3}(H_J \times H_{-I})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho_0^\infty, \zeta_0^\infty)[h] = \frac{1}{2} \mathbb{E}\left[D^2 F(\zeta_t^\infty) : DV_t(\rho_0^\infty)[h] \right]$$

with

$$V_t(\rho_0^\infty)(\varphi, \psi) = \mathrm{Cov}\left(\langle \varphi, \zeta_t^\infty \rangle, \langle \psi, \zeta_t^\infty \rangle \right)$$
$$= \frac{1}{2} \int_0^t \left\langle \nabla P_{t-s}^{HE} \varphi \cdot \nabla P_{t-s}^{HE} \psi, \rho_s^\infty (1-\rho_s^\infty) \right\rangle ds$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}F(\rho_t^{\infty,n},\zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^{\infty},\zeta_t^{\infty}) = P_t^{OU}F(\hat{\rho}_0^n,\hat{\zeta}_0^n) - P_t^{OU}F(\rho_0,\zeta_0)$$

where ρ_t^∞ started from the initial profile ρ_0 and ζ_t started from the centered Gaussian distribution with

$$\mathbb{E}\langle\zeta_0,\varphi\rangle^2 = \langle\rho_0(1-\rho_0)\varphi,\varphi\rangle.$$

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where $G \in \mathrm{C}^3(H_{-l})$, where

$$\mathrm{ex}_n\zeta_0^n := \sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0^n,\varsigma_k\rangle_n\varsigma_k, \quad \mathrm{pr}_n\zeta_0 := \sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0,\varsigma_k\rangle\varsigma_k$$

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- Is enough to compare for $g \in \mathrm{C}^3\left(\mathbb{R}^{\mathbb{Z}_n^d}\right)$
$$\mathbb{E}g\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0^n,\varsigma_k\rangle_n\right)_{k\in\mathbb{Z}_n^d}\right) - \mathbb{E}g\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0,\varsigma_k\rangle\right)_{k\in\mathbb{Z}_n^d}\right).$$

Berry-Esseen bound for the initial fluctuations

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$$\mathbb{E}F(\rho_t^{\infty,n},\zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^{\infty},\zeta_t^{\infty}) = P_t^{OU}F(\hat{\rho}_0^n,\hat{\zeta}_0^n) - P_t^{OU}F(\rho_0,\zeta_0)$$

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- Apply multidimensional Berry-Essen theorem [e.g., Meckes '09]

References

[1] Benjamin Gess and Vitalii Konarovskyi. A quantitative central limit theorem for the simple symmetric exclusion process (2024), arXiv:2408.01238

Thank you!