A Central Limit Theorem for Modified Massive Arratia Flow

Vitalii Konarovskyi

University of Hamburg and Institute of Mathematics of NAS of Ukraine

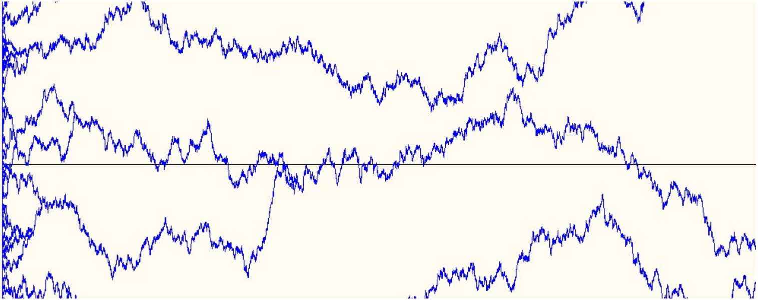
Malliavin Calculus and its Applications

joint work with Andrey Dorogovtsev and Max von Renesse

Coalescing particle system: Arratia flow

Arratia flow on $\mathbb{R}$ [R. Arratia '79]

- Brownian particles start from every point of an interval or real line;
- they move independently and coalesce after meeting;



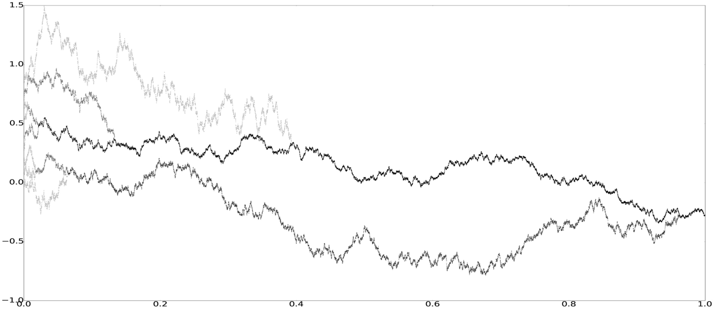
Arratia flow and its generalization

- **Arratia flow appears as scaling limit of different models**
  - true self-repelling motion [B.Tóth and W. Werner (PTRF '98)]
  - isotropic stochastic flows of homeomorphisms in $\mathbb{R}$ [V. Piterbarg (Ann. Prob. '98)]
  - Hastings-Levitov planer aggregation models [J. Norris, A. Turner (Comm. Math. Phys. '12)], etc...
- **Further investigation of the Arratia flow**
  - Properties of generated σ-algebra [B. Tsirelson (Probab. Surv. '04)]
  - n-particle motion [R. Tribe, O.V. Zaboronski (EJP '04, Comm. Math. Phys. '06)]
  - large deviations [A. Dorogovtsev, O. Ostapenko (Stoch. Dyn. '10)], etc...
- **Generalizations**
  - Brownian web [C. M. Newman et al. (Ann. Prob. '04), R. Sun, J.M Swart (MAMS, '14)]
  - Coalescing non-Brownian particles [S. Evans et al. (PTRF, '13)]
  - Stochastic flows of kernels [Y. Le Jan and O. Raimond (Ann. Prob. '04)]

Modified Massive Arratia flow (MMAF)

Modified massive Arratia flow on $\mathbb{R}$ [K. (Ann. Prob. '17, EJP '17)]

- Brownian particles start from points with masses;
- they move independently and coalesce after meeting;
- particles sum their masses after meeting and diffusion rate is inversely proportional to the mass.



Mathematical description and properties

Mathematical description

Let $X(u,t)$ is the position of particle at time t labeled by u

1. $X(u,0) = u$;
2. $X(u,\cdot)$ is a continuous martingale;
3. $X(u,t) \leq X(v,t)$, $u < v$;
4. $\langle X(u,\cdot), X(v,\cdot)\rangle_{t\wedge\tau_{u,v}} = 0$, where $\tau_{u,v} = \inf\{t : \ X(u,t) = X(v,t)\}$;
5. $\langle X(u,\cdot)\rangle_t = \int_0^t \frac{1}{m(u,s)} ds$, where $m(u,s)$ is the mass of part. u at time s

Mathematical description and properties

Mathematical description

Let $X(u,t)$ is the position of particle at time t labeled by u

1. $X(u,0)=u$;
2. $X(u,\cdot)$ is a continuous martingale;
3. $X(u,t)\leq X(v,t)$, $u<v$;
4. $\langle X(u,\cdot),X(v,\cdot)\rangle_{t\wedge\tau_{u,v}}=0$, where $\tau_{u,v}=\inf\{t:\ X(u,t)=X(v,t)\}$;
5. $\langle X(u,\cdot)\rangle_t=\int_0^t\frac{1}{m(u,s)}ds$, where $m(u,s)$ is the mass of part. u at time s

Connection with Dean-Kawasaki eq. and Wasserstein diff. [K., Renesse, CPAM '19]

The process μ_t, $t\geq 0$, that describes the evolution of particle masses solves

$$d\mu_t=\frac{1}{2}\Delta\mu_t^*dt+\nabla\cdot(\sqrt{\mu_t}dW_t),$$

and satisfies Varadhan's formulat

$$\mathbb{P}\{\mu_t=\nu\}\sim e^{-\frac{\mathcal{W}_2^2(\mu_0,\nu)}{2t}},\quad t\to 0+,$$

with Wasserstein distance $\mathcal{W}_2$ in $\mathcal{P}_2(\mathbb{R})$.

MMAF started from integer points

Let $\{X_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ be a family of processes such that

1. X_k is a continuous square-integrable martingale with respect to the joint filtration;
2. $X_k(0) = k$;
3. $X_k(t) \leq X_l(t)$ for $k < l$;
4. $\langle X_k, X_l \rangle_{t \wedge \tau_{k,l}} = 0$, where $\tau_{k,l} = \inf\{t :\ X_k(t) = X_l(t)\}$;
5. $\langle X_k \rangle_t = \int_0^t \frac{ds}{m_k(s)}$, where $m_k(t) = \#\{l :\ \exists s \leq t\ \ X_k(s) = X_l(s)\}$;

Theorem [K (TVP '10)]

There exists a family of stochastic processes $\{X_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ satisfying the assumptions 1.-5. Moreover, the assumptions 1.-5. uniquely determine the distribution in $\mathrm{C}[0, \infty)^{\mathbb{Z}}$.

CLT for occupation measure

- We define the **occupation measure** defined by

$$N_t(A) = \#\left(A \cap \{X_k(t), k \in \mathbb{Z}\}\right), \quad A \in \mathcal{B}(\mathbb{R}).$$

- Let $\mathcal{P}$ denote the set of bounded measurable one-periodic functions $f : \mathbb{R} \to \mathbb{R}$;
- For $f \in \mathcal{P}$ set

$$A_{k,t} f := \int_{k-1}^{k} f(u) N_t(du).$$

Theorem [Dorogovtsev, K., von Renesse '24]

For every $f \in \mathcal{P}$ and $t > 0$

$$Y_t^n(f) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(A_{k,t} f - \mathbb{E}\left[A_{k,t} f\right]\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_t^2(f)\right)$$

with

$$\sigma_t^2(f) = \operatorname{Var} A_{0,t} f + 2 \sum_{k=1}^{\infty} \operatorname{Cov}(A_{0,t} f, A_{k,t} f)\,.$$

Comparison with similar result for Arratia flow

Let $\tilde{N}_t$ be the occupation measure for the Arratia flow, $\tilde{A}_{k,t}$ be defined similarly for $\tilde{N}_t$

Theorem [Dorogovtsev, Hlyniana (Stoch. and Dyn. '23)]

For every $f \in \mathcal{P}$ and $t > 0$

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(\tilde{A}_{k,t} f - \mathbb{E} \left[\tilde{A}_{k,t} f \right] \right) \xrightarrow{d} \mathcal{N} \left(0, \tilde{\sigma}_t^2(f) \right)$$

with

$$\tilde{\sigma}_t^2(f) = \mathrm{Var}\, A_{0,t} f + 2 \sum_{k=1}^{\infty} \mathrm{Cov}(A_{0,t} f, A_{k,t} f)$$

$$= \frac{1}{\sqrt{\pi t}} \int_0^1 f^2(u) du + \int_0^1 \int_0^1 f(u) f(v) G_t(u,v) du dv,$$

with

$$G_t(u,v) = g_t(u-v) + 2 \sum_{k=1}^{\infty} g_t(u-v+k), \quad g_t(u-v) = \rho_t^{(2)}(u,v) - \frac{1}{\pi t}.$$

Strategy of proof of both results

The proofs are based on the classical CLT for stationary sequences

(e.g. [Ibragimov, Linnik '71])

Theorem

Let ξ_k be a stationaly sequence satisfying the strong mixing condition with mixing coefficient

$$\alpha(n) := \sup_{A\in\mathfrak{M}_{-\infty}^{0},B\in\mathfrak{M}_{n}^{+\infty}} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| \to 0$$

and $\mathbb{E}\left[|\xi_k|^{2+\delta}\right] < \infty$. If $\sum_{n=1}^{\infty} \alpha(n)^{\frac{\delta}{2+\delta}} < \infty$, then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\xi_k - \mathbb{E}[\xi_k]) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

with

$$\sigma^2 = \mathrm{Var}\,\xi_0 + 2\sum_{k=1}^{\infty} \mathrm{Cov}(\xi_0, \xi_k)\,.$$

Strong mixing condition for MMAF

We set for $f \in \mathcal{P}$ and $t > 0$

$$\alpha_i(j) := \sup_{A \in \mathfrak{M}^i_{-\infty}, B \in \mathfrak{M}^{\infty}_j} \left|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\right|, \quad j > i,$$

where

$$\mathfrak{M}^b_a = \sigma\left\{A_{k,t}f,\ a \leq k \leq b\right\}.$$

Strong mixing condition for MMAF

We set for $f \in \mathcal{P}$ and $t > 0$

$$\alpha_i(j) := \sup_{A \in \mathfrak{M}_{-\infty}^i, B \in \mathfrak{M}_j^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, \quad j > i,$$

where

$$\mathfrak{M}_a^b = \sigma\{A_{k,t}f, \ a \leq k \leq b\}.$$

Proposition [Dorogovtsev, K., von Renesse '24]

There exist a constant $C > 0$ and $\beta > 0$ depending only on f and t such that

$$\alpha_i(j) \leq Ce^{-\beta\sqrt{j-i}}$$

for all $i < j$.

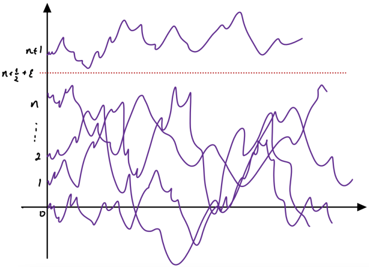
Gap property for independent Brownian motions

Lemma [K., (TVP '10)]

Let w_k, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a family of independent Brownian motions on $\mathbb{R}$ with diffusion rate 1 and $w_k(0) = k$. Then for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ the equality

$$\mathbb{P}\left\{ \max_{k \in \{0,\dots,n\}} \max_{t \in [0,T]} w_k(t) \le n + \frac{1}{2}, \quad \min_{t \in [0,T]} w_{n+1}(t) > n + \frac{1}{2} + \varepsilon \text{ i.o.} \right\} = 1$$

holds.



Construction of MMAF

We construct the process $\{X_k^n\}_{k=-n,\ldots,+n}$ from the family of independent Brownian motions by coalescing their paths by a special way:

Construction of MMAFs for $\mathbb{Z}_{\geq l}$ and $\mathbb{Z}_{\leq l}$

We similarly construct the processes $\left\{X_k^{n,+}\right\}_{k=0,\ldots,n}$ and $\left\{X_k^{n,-}\right\}_{k=-n,\ldots,1}$ from the same family of independent Brownian motions:

Passing to the limit

Lemma

1. The process X_k^n converges a.s. in the discrete topology of $\mathrm{C}[0,T]$ to a process X_k as $n \to \infty$ for each $k \in \mathbb{Z}$, where $\{X_k\}_{k\in\mathbb{Z}}$ as the MMAF started from $\mathbb{Z}$.
2. The process $X_k^{n,+}$ converges a.s. in the discrete topology of $\mathrm{C}[0,T]$ to a process X_k^+ as $n \to \infty$ for each $k \in \mathbb{Z}_{\geq 0}$, where $\left\{X_k^+\right\}_{k\in\mathbb{Z}_{\geq 0}}$ as the MMAF started from $\mathbb{Z}_{\geq 0}$.
3. The same for $X_k^{n,-}$.

Control of probability of appearing of gaps

We set

$$A_j^+(t) := \left\{ \max_{k\in\{0,\dots,j\}} \max_{s\in[0,t]} w_k(s) \le j + \frac{1}{2}, \quad \min_{s\in[0,t]} w_{j+1}(s) > j + \frac{1}{2} \right\}$$

and

$$A_j^-(t) := \left\{ \min_{k\in\{-j,\dots,-1\}} \min_{s\in[0,t]} w_k(s) \ge -j - \frac{1}{2}, \quad \max_{s\in[0,t]} w_{-j-1}(s) < -j - \frac{1}{2} \right\}$$

for all $j \in \mathbb{N}$ and $t \in [0, T]$.

Control of probability of appearing of gaps

We set

$$A_j^+(t) := \left\{ \max_{k\in\{0,\dots,j\}} \max_{s\in[0,t]} w_k(s) \le j + \frac{1}{2}, \quad \min_{s\in[0,t]} w_{j+1}(s) > j + \frac{1}{2} \right\}$$

and

$$A_j^-(t) := \left\{ \min_{k\in\{-j,\dots,-1\}} \min_{s\in[0,t]} w_k(s) \ge -j - \frac{1}{2}, \quad \max_{s\in[0,t]} w_{-j-1}(s) < -j - \frac{1}{2} \right\}$$

for all $j \in \mathbb{N}$ and $t \in [0, T]$.

We also define

$$B_N^+(t) = \bigcup_{j=1}^{N} A_j^+(t) \quad \text{and} \quad B_N^-(t) = \bigcup_{k=1}^{N} A_j^-(t).$$

Control of probability of appearing of gaps

We set

$$A_j^+(t) := \left\{ \max_{k\in\{0,\dots,j\}} \max_{s\in[0,t]} w_k(s) \le j + \frac{1}{2}, \quad \min_{s\in[0,t]} w_{j+1}(s) > j + \frac{1}{2} \right\}$$

and

$$A_j^-(t) := \left\{ \min_{k\in\{-j,\dots,-1\}} \min_{s\in[0,t]} w_k(s) \ge -j - \frac{1}{2}, \quad \max_{s\in[0,t]} w_{-j-1}(s) < -j - \frac{1}{2} \right\}$$

for all $j \in \mathbb{N}$ and $t \in [0, T]$.

We also define

$$B_N^+(t) = \bigcup_{j=1}^{N} A_j^+(t) \quad \text{and} \quad B_N^-(t) = \bigcup_{k=1}^{N} A_j^-(t).$$

Remark. $B_N^+(t)$ means that there is a gap between 0 and N of length t.

Control of probability of appearing of gaps

Proposition

For each $T > 0$ there exist a constant $C = C_T > 0$ and a function $\beta_T(t) : (0, T] \to (0, \infty)$ depending only on T such that $t\beta_T(t) \to \frac{1}{8\sqrt{2}}$ as $t \to 0+$ and for every $N \in \mathbb{N}$

$$\mathbb{P}\left(B_N^{\pm}(t)\right) \geq 1 - Ce^{-\beta_T(t)\left[(\sqrt{N}-\sqrt{2})\vee 1\right]}$$

for all $N \in \mathbb{N}$ and $t \in [0, T]$.

Control of probability of appearing of gaps

Proposition

For each $T > 0$ there exist a constant $C = C_T > 0$ and a function $\beta_T(t) : (0, T] \to (0, \infty)$ depending only on T such that $t\beta_T(t) \to \frac{1}{8\sqrt{2}}$ as $t \to 0+$ and for every $N \in \mathbb{N}$

$$\mathbb{P}\left(B_N^{\pm}(t)\right) \geq 1 - Ce^{-\beta_T(t)\left[(\sqrt{N}-\sqrt{2})\vee 1\right]}$$

for all $N \in \mathbb{N}$ and $t \in [0, T]$.

Lemma

For each $k \geq N$ the processes $(X_k(t))_{t\in[0,T]}$ and $\left(X_k^+(t)\right)_{t\in[0,T]}$ coincide on $B_N^+(t)$ and $(X_{-k}(t))_{t\in[0,T]}$ and $\left(X_{-k}^-(t)\right)_{t\in[0,T]}$ coincide on $B_N^-(t)$.

Idea of control of mixing coefficient

We recall

$$\alpha_i(j) := \sup_{A \in \mathfrak{M}^i_{-\infty}, B \in \mathfrak{M}^{\infty}_j} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, \quad j > i,$$

where $\mathfrak{M}^b_a = \sigma\{A_{k,t}f,\ a \leq k \leq b\}$, $A_{k,t}f := \int_{k-1}^{k} f(u) N_t(du)$ and $N_t(A) = \#\left(A \cap \{X_k(t), k \in \mathbb{Z}\}\right)$.

Idea of control of mixing coefficient

We recall

$$\alpha_i(j) := \sup_{A \in \mathfrak{M}^i_{-\infty}, B \in \mathfrak{M}^\infty_j} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, \quad j > i,$$

where $\mathfrak{M}^b_a = \sigma\{A_{k,t}f,\ a \leq k \leq b\}$, $A_{k,t}f := \int_{k-1}^k f(u)N_t(du)$ and $N_t(A) = \#(A \cap \{X_k(t), k \in \mathbb{Z}\})$.

We also define

$$A^\pm_{k,t}f = \int_{k-1}^k f(u)N^\pm_t(du)$$

for $N^+_t(A) = \#\left(A \cap \left\{X^+_k(t), k \in \mathbb{Z}_{\geq 0}\right\}\right)$ and $N^-_t(A) = \#\left(A \cap \left\{X^-_k(t), k \in \mathbb{Z}_{\leq -1}\right\}\right)$

Idea of control of mixing coefficient

Without loss of generality we assume that $j \in \mathbb{Z}_{\geq 0}$ and $i = -j$.

Idea of control of mixing coefficient

Without loss of generality we assume that $j \in \mathbb{Z}_{\geq 0}$ and $i = -j$.

For $A \in \mathfrak{M}^{i}_{-\infty}$ and $B \in \mathfrak{M}^{+\infty}_{j}$ there exist Borel measurable sets $\tilde{A} \subseteq \mathbb{R}^{\mathbb{Z}_{\leq i}}$ and $\tilde{B} \subseteq \mathbb{R}^{\mathbb{Z}_{\geq j}}$ such that

$$A = \left\{ (A_{k,t} f)_{k \in \mathbb{Z}_{\leq i}} \in \tilde{A} \right\}, \quad B = \left\{ (A_{k,t} f)_{k \in \mathbb{Z}_{\geq j}} \in \tilde{B} \right\}.$$

Idea of control of mixing coefficient

Without loss of generality we assume that $j \in \mathbb{Z}_{\geq 0}$ and $i = -j$.

For $A \in \mathfrak{M}_{-\infty}^{i}$ and $B \in \mathfrak{M}_{j}^{+\infty}$ there exist Borel measurable sets $\tilde{A} \subseteq \mathbb{R}^{\mathbb{Z}_{\leq i}}$ and $\tilde{B} \subseteq \mathbb{R}^{\mathbb{Z}_{\geq j}}$ such that

$$A = \left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\}, \quad B = \left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\}.$$

$$\begin{aligned}
|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| &= \Big| \mathbb{P}\left(\left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\} \cap \left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\} \right) \\
&\qquad - \mathbb{P}\left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\} \mathbb{P}\left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\} \Big| \\
&\leq \Big| \mathbb{P}\left(\left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\} \cap \left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\} \cap B_j^+(t) \cap B_{-i}^-(t) \right) \\
&\qquad - \mathbb{P}\left(\left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\} \cap B_{-i}^- \right) \mathbb{P}\left(\left\{ (A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\} \cap B_j^+ \right) \Big| \\
&+ Ce^{-\beta T(t)\sqrt{j-i}} \\
&= \Big| \mathbb{P}\left(\left\{ (A_{k,t}^-f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\} \cap \left\{ (A_{k,t}^+f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\} \cap B_j^+ \cap B_{-i}^- \right) \\
&\qquad - \mathbb{P}\left(\left\{ (A_{k,t}^-f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A} \right\} \cap B_{-i}^- \right) \mathbb{P}\left(\left\{ (A_{k,t}^+f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B} \right\} \cap B_j^+ \right) \Big| \\
&+ Ce^{-\beta T(t)\sqrt{j-i}}.
\end{aligned}$$

Positivity of $\sigma_t^2(f)$

Proposition

Let $f \in \mathrm{C}_b^3(\mathbb{R})$ be an odd, 1-periodic function. Then $\frac{\sigma_t^2(f)}{t} \to (f'(0))^2$ as $t \to 0+$. In particular, there exists $t > 0$ such that $\sigma_t^2(f) > 0$ if $f'(0) \neq 0$.

Positivity of $\sigma_t^2(f)$

Proposition

Let $f \in \mathrm{C}_b^3(\mathbb{R})$ be an odd, 1-periodic function. Then $\frac{\sigma_t^2(f)}{t} \to (f'(0))^2$ as $t \to 0+$. In particular, there exists $t > 0$ such that $\sigma_t^2(f) > 0$ if $f'(0) \neq 0$.

Idea of Proof. We define

$$\tilde{A}_{k,t} f := \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u) N_t(du).$$

Positivity of $\sigma_t^2(f)$

Proposition

Let $f \in \mathrm{C}_b^3(\mathbb{R})$ be an odd, 1-periodic function. Then $\frac{\sigma_t^2(f)}{t} \to (f'(0))^2$ as $t \to 0+$. In particular, there exists $t > 0$ such that $\sigma_t^2(f) > 0$ if $f'(0) \neq 0$.

Idea of Proof. We define

$$\tilde{A}_{k,t} f := \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u) N_t(du).$$

Note that $\mathbb{E}\left[\tilde{A}_{k,t} f\right] = 0$ and define

$$\begin{aligned} \tilde{Y}_t^n(f) &:= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(\tilde{A}_{k,t} f - \mathbb{E}\left[\tilde{A}_{k,t} f\right]\right) \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \tilde{A}_{k,t} f = \frac{1}{\sqrt{n}} \int_{\frac{1}{2}}^{n+\frac{1}{2}} f(u) N_t(du) \end{aligned}$$

Idea of proof

Thus

$$\mathbb{E}\left[\left(Y_t^n(f) - \tilde{Y}_t^n(f)\right)^2\right] \le \frac{2}{n}\mathbb{E}\left[\left(\int_0^{\frac{1}{2}} f(u)N_t(du)\right)^2\right] + \frac{2}{n}\mathbb{E}\left[\left(\int_n^{n+\frac{1}{2}} f(u)N_t(du)\right)^2\right] \to 0.$$

Idea of proof

Thus

$$\mathbb{E}\left[\left(Y_t^n(f) - \tilde{Y}_t^n(f)\right)^2\right] \leq \frac{2}{n}\mathbb{E}\left[\left(\int_0^{\frac{1}{2}} f(u)N_t(du)\right)^2\right] + \frac{2}{n}\mathbb{E}\left[\left(\int_n^{n+\frac{1}{2}} f(u)N_t(du)\right)^2\right] \to 0.$$

As before, we can proof

$$\tilde{Y}_t^n(f) \to \mathcal{N}(0, \tilde{\sigma}_t^2),$$

where

$$\tilde{\sigma}_t^2 = \mathrm{Var}\,\tilde{A}_{0,t}f + 2\sum_{k=1}^{\infty} \mathrm{Cov}\left(\tilde{A}_{0,t}f, \tilde{A}_{k,t}f\right) = \mathbb{E}\left[\left(\tilde{A}_{0,t}f\right)^2\right] + 2\sum_{k=1}^{\infty}\mathbb{E}\left[\tilde{A}_{0,t}f\tilde{A}_{k,t}f\right].$$

Idea of proof

Set

$$B := \left\{ |X_0(t)| \leq \frac{1}{2} \right\} \cap \left\{ X_{-1}(t) \leq -\frac{1}{2} \right\} \cap \left\{ X_1(t) \geq \frac{1}{2} \right\}$$

Then

$$\begin{aligned} \mathbb{E}\left[\left(\tilde{A}_{0,t} f \right)^2 \right] &= \mathbb{E}\left[f^2(w_0(t)) \right] \\ &+ \mathbb{E}\left[\left(\left(\tilde{A}_{0,t} f \right)^2 - f^2(w_0(t)) \right) \mathbb{I}_{B^c} \right] \\ &= f^2(0) + \frac{1}{2} \frac{d^2 f^2}{dx^2}(0) + \mathbb{E}\left[w_0^2(t) \right] + o(t) \\ &= (f'(0))^2 t + o(t). \end{aligned}$$

Idea of proof

Set

$$B := \left\{ |X_0(t)| \le \frac{1}{2} \right\} \cap \left\{ X_{-1}(t) \le -\frac{1}{2} \right\} \cap \left\{ X_1(t) \ge \frac{1}{2} \right\}$$

Then

$$\begin{aligned} \mathbb{E}\left[\left(\tilde{A}_{0,t} f \right)^2 \right] &= \mathbb{E}\left[f^2(w_0(t)) \right] \\ &+ \mathbb{E}\left[\left(\left(\tilde{A}_{0,t} f \right)^2 - f^2(w_0(t)) \right) \mathbb{I}_{B^c} \right] \\ &= f^2(0) + \frac{1}{2} \frac{d^2 f^2}{dx^2}(0) + \mathbb{E}\left[w_0^2(t) \right] + o(t) \\ &= (f'(0))^2 t + o(t). \end{aligned}$$

Using the lemma about gaps, we get

$$\mathbb{E}\left[\tilde{A}_{0,t} f \tilde{A}_{k,t} f \right] \le C_T e^{-\frac{\beta_T(t)}{2}[(\sqrt{k}-\sqrt{2}) \vee 1]}$$

with $t\beta_T(t) \to \frac{1}{8\sqrt{2}}$ as $t \to 0+$. Thus,

$$\frac{1}{t} \sum_{k=0}^{\infty} \left| \mathbb{E}\left[\tilde{A}_{0,t} f \tilde{A}_{k,t} f \right] \right| \to 0, \quad t \to 0+.$$

Idea of proof

Consequently,

$$\frac{1}{t}\tilde{\sigma}_t^2 = (f'(0))^2 + \frac{o(t)}{t} + \frac{2}{t}\sum_{k=0}^{\infty} \mathbb{E}\left[\tilde{A}_{0,t} f \tilde{A}_{k,t} f\right] \to (f'(0))^2.$$

References

[1] Andery Dorogovtsev, Vitalii Konarovskyi, Max von Renesse. A Central Limit Theorem for Modified Massive Arratia Flow (2024), *arXiv:2408.05030*

[2] Vitalii Konarovskyi, Max von Renesse. Modified Massive Arratia flow and Wasserstein diffusion (2019), *Communications on Pure and Applied Mathematics*

[3] Vitalii Konarovskyi. A system of coalescing heavy diffusion particles on the real line (2017), *Annals of Probability*

[4] Vitalii Konarovskyi. On infinite system of diffusion particles with coalescing (2011), *Theory of Probability and Its Applications*

Thank you!