Coalescing-fragmentating Wasserstein dynamics: particle approach

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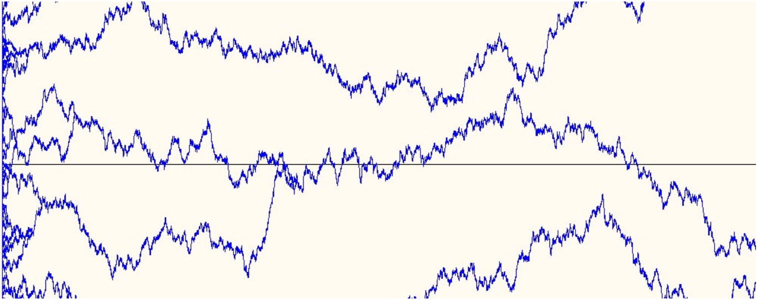
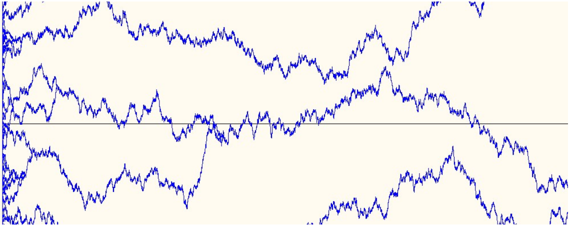
Coalescing particle system: Arratia flow

Arratia flow on $\mathbb{R}$ (R. Arratia '79)

- Brownian particles start from every point of an interval;
- they move independently and coalesce after meeting;



Mathematical description of Arratia flow



$X(u,t)$ is the position of particle at time t starting at u

1. $X(u,0) = u$;
2. $X(u,\cdot)$ is a Brownian motion in $\mathbb{R}$;
3. $X(u,t) \leq X(v,t)$, $u < v$
4. $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \mathbb{I}_{\{X(u,s)=X(v,s)\}} ds$.

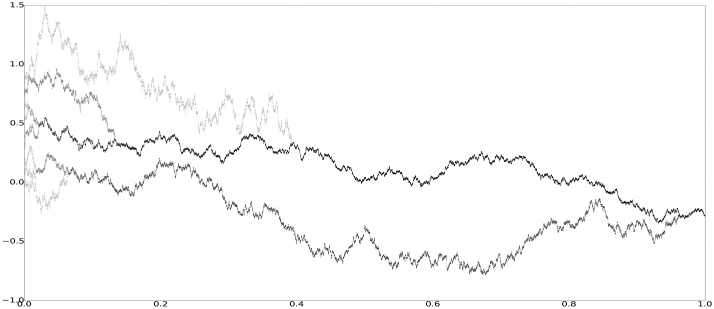
Arratia flow and its generalization

- **Arratia flow appears as scaling limit of different models**
    - true self-repelling motion (B.Tóth and W. Werner (PTRF '98))
    - isotropic stochastic flows of homeomorphisms in $\mathbb{R}$ (V. Piterbarg (Ann. Prob. '98))
    - Hastings-Levitov planer aggregation models (J. Norris, A. Turner (Comm. Math. Phys. '12)), etc...
- **Further investigation of the Arratia flow**
    - Properties of generated σ-algebra (B. Tsirelson (Probab. Surv. '04))
    - n-particle motion (R. Tribe, O.V. Zaboronski (EJP '04, Comm. Math. Phys. '06))
    - large deviations (A. Dorogovtsev, O. Ostapenko (Stoch. Dyn. '10)), etc...
- **Generalizations**
    - Brownian web (C. M. Newman et al. (Ann. Prob. '04), R. Sun, J.M Swart (MAMS, '14))
    - Coalescing non-Brownian particles (S. Evans et al. (PTRF, '13))
    - Stochastic flows of kernels (Y. Le Jan and O. Raimond (Ann. Prob. '04))

Modified Massive Arratia flow

Modified massive Arratia flow on $\mathbb{R}$ (K. '17)

- Brownian particles start from points with masses;
- they move independently and coalesce after meeting;
- particles sum their masses after meeting and diffusion rate is inversely proportional to the mass.



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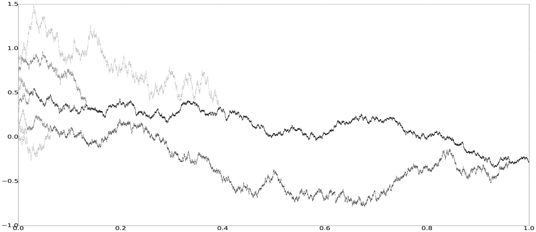

$Y(u,t)$ is the position of particle at time t labeled by $u \in (0,1)$

1. $Y(u,0) = u$;
2. $Y(u,\cdot)$ is a continuous martingale;
3. $Y(u,t) \le Y(v,t)$, $u < v$;
4. $\langle Y(u,\cdot), Y(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{Y(u,s)=Y(v,s)\}}}{m(u,s)} ds$,
   $m(u,s) = \mathrm{Leb}\{w : \ Y(w,s) = Y(u,s)\}$.

Short-time asymptotic of a Brownian motion

Short-time asymptotic formula for a heat kernel

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{2t}} \sim e^{-\frac{\|x-y\|^2}{2t}}, \quad t \to 0+ \,.$$

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Generalizations

- Heat equation with variable coefficients in $\mathbb{R}^n$ (Varadhan (CPAM '67))
- Smooth Riemannian manifold with Ricci curvature bound (P. Li and S.-T. Yau (Acta Math. '86))
- Lipschitz Riemannian manifold without any sort of curvature bounds (J. Norris (Acta Math. 97))
- Infinite-dimensional case for heat kernel generated by a Dirichlet form (J. Ramírez (CPAM '01, Ann. Prob '03))

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Corollary

If B_t, $t \geq 0$, is a Brownian motion on a Riemannian manifold, then

$$\mathbb{P}_x\{B_t = y\} \sim e^{-\frac{d^2(x,y)}{2t}}, \quad t \to 0+,$$

with d being the Riemannian distance.

Connection with optimal transport

Theorem (K./ Renesse, '19)

The process $\mu_t = Y(\cdot, t)\big|_{\#}\text{Leb}$, $t \geq 0$, which describes the evolution of particle masses in the modified massive Arratia flow satisfies Varadhan's formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0, \nu)}{2t}}, \quad t \to 0+,$$

with the quadratic Wasserstein distance $d_{\mathcal{W}}$ in $\mathbb{R}$.

Quadratic Wasserstein distance:

$$d_{\mathcal{W}}(\nu_1, \nu_2) = \inf_{\xi_1 \sim \nu_1, \xi_2 \sim \nu_2} \left(\mathbb{E}|\xi_1 - \xi_2|^2\right)^{\frac{1}{2}}$$

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$(\mathcal{P}_2(\mathbb{R}), d_{\mathcal{W}})$ has an inf.-dim. Riemannian structure (F. Otto (JFA, '01)).

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Sticky-reflected interaction

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Remind that the coalescing particle system X satisfies the following properties:

1. $X(u,0) = u$, $u \in [0,1]$
2. $X(u,\cdot)$ is a continuous martingale
3. $X(u,t) \le X(v,t)$, $u < v$;
4. $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$,
   $m(u,s) = \mathrm{Leb}\{w : X(w,t) = X(u,t)\}$.

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Sticky-reflected interaction

Can we replace the coalescing by another type of interaction to have the same Varadhan formula and get a dynamics which is reversible in time?

Remind that the coalescing particle system X satisfies the following properties:

1. $X(u,0)=g(u)$, $u\in[0,1]$, where $g\uparrow$;
2. $X(u,\cdot)$ is a continuous martingale
3. $X(u,t)\leq X(v,t)$, $u<v$;
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   $m(u,s)=\mathrm{Leb}\{w: X(w,t)=X(u,t)\}$.

$X(u,t)$ is the position of particle at time t started from $g(u)$
(initial particle distribution$=\mathrm{Leb}\circ g^{-1}$) .

Sticky-reflected interaction

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Remind that the coalescing particle system X satisfies the following properties:

1. $X(u,0) = g(u)$, $u \in [0,1]$, where $g \uparrow$;
2. $X(u,\cdot) - \int_0^t \left(\xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(r)dr\right) ds$ is a continuous martingale, where $\pi(u,t) = \{v : \ X(u,t) = X(v,t)\}$ and $\xi \uparrow$ is the **interaction potential;**
3. $X(u,t) \le X(v,t)$, $u < v$;
4. $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$, $m(u,s) = \mathrm{Leb}\{w : X(w,t) = X(u,t)\}$.

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Role of function ξ

Remind that $X(u,\cdot) - \int_0^t \left(\xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(r)dr\right) ds$ is a continuous martingale, where $\pi(u,t) = \{v : \ X(u,t) = X(v,t)\}$ and $\xi \uparrow$.

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- If $\xi(u) = \xi(v)$, then particles u and v coalesce after the meeting:
  because the drifts of $X(u,\cdot)$ and $X(v,\cdot)$ at time s are equal after the meeting

$$\xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(u)du = \xi(v) - \frac{1}{m(v,s)} \int_{\pi(v,s)} \xi(r)dr,$$

  since $\pi(u,s) = \pi(v,s)$ for $X(u,s) = X(v,s)$.

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$\rightsquigarrow$ If $g(u) = g(v)$, $\xi(u) = \xi(v)$, then $X(u,\cdot) = X(v,\cdot)$.

$\rightsquigarrow$ If $g = \sum_{i=1}^n x_i \mathbb{I}_{\pi_i}$, $\xi = \sum_{i=1}^n \xi_i \mathbb{I}_{\pi_i}$, then

$$X(u,t) = \sum_{i=1}^n x_i(t)\mathbb{I}_{\pi_i}(u).$$

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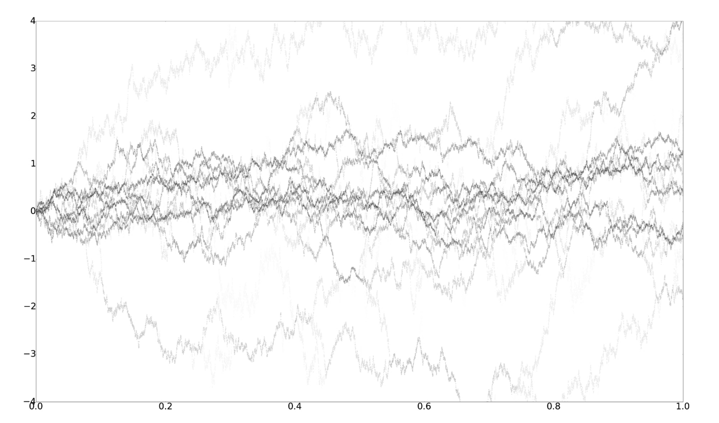

$$g(u) = 0, \quad \xi(u) = u, \quad u \in (0,1)$$

The model is similar to the Howitt-Warren flow. The main difference is that in our case particles change the diffusion rate.

(Howitt, Warren '09; Schertzer, Sun, Swart '14)

Existence of the particle system

Theorem

Let $g, \xi : [0,1] \to \mathbb{R}$ be non-decreasing $\frac{1}{2}+$-Hölder continuous functions. Then there exists a random càdlàg map $[0,1] \ni u \mapsto X(u,\cdot) \in C[0,\infty)$ such that

1. $X(u,0) = g(u)$
2. $X(u,\cdot) - \int_0^t \left(\xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(r)dr\right) ds$ is a continuous martingale, where $\pi(u,t) = \{v : \ X(u,t) = X(v,t)\}$, $m(u,s) = \mathrm{Leb}\{w : X(w,t) = X(u,t)\}$;
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Uniqueness is an open problem

Idea of construction

- For $g^n = \sum_{i=1}^n x_i^0 \mathbb{I}_{\pi_i}$, $\xi^n = \sum_{i=1}^n \xi_i \mathbb{I}_{\pi_i}$, for an ordered partition $\{\pi_i\}$ of $[0,1]$

$$X_n(u,t) = \sum_{i=1}^n x_i(t) \mathbb{I}_{\pi_i}(u).$$

$\rightsquigarrow$ existence of $\{x_i\}$ is obtained by solving of a corresponding SDE in $\mathbb{R}^n$.

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- A priori estimate

$$\int_0^t \mathbb{P}\{m(u,s) < r\}\, ds \leq C_t r \left[(g(u \pm r) - g(u))^2 + (\xi(u \pm r) - \xi(u))^2 \right].$$

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- Tightness of finite particle system if $g^n \to g$, $\xi^n \to \xi$.

SDE in $L_2^{\uparrow}$ for the particle system

There exists a space time white noise such that

$$dX(u,t) = \frac{1}{m(u,t)} \int_{\pi(u,t)} W(dr,dt) + \left(\xi(u) - \frac{1}{m(u,t)} \int_{\pi(u,t)} \xi(r)dr \right) dt.$$

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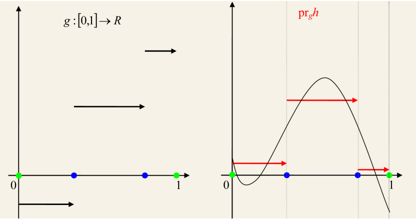
Let pr_g be the **projection** in $L_2[0,1]$ onto

$$L_2(g) = \{f : f \text{ is } \sigma(g)\text{-measurable}\}$$



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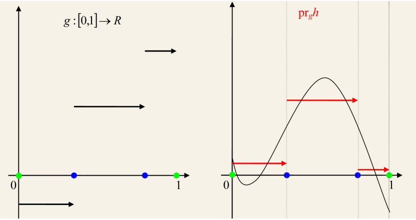

Then $X_t := X(\cdot,t) \in L_2^{\uparrow}$ solves

$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t} \xi)dt.$$

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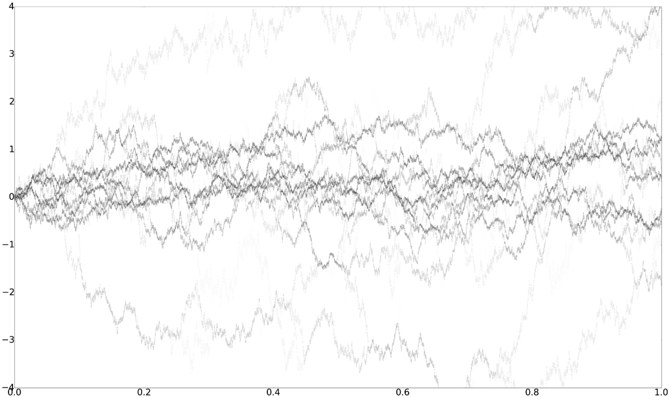
Number of particles

How many distinct particles does the system contain at every time t?



$$g(u) = 0, \quad \xi(u) = u, \quad u \in (0, 1)$$

Finite number of particles

$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t} \xi)dt.$$

Hence, for the martingale part M of X we have

$$\mathbb{E}\|M_t\|_t^2 = \int_0^t \mathbb{E}\| \mathrm{pr}_{X_s} \|_{HS}^2 ds = \int_0^t \mathbb{E}N(s)ds < \infty,$$

where $N(t)$ is the number of distinct particles at time t.

Infinite number of particles

Theorem

Let ξ takes infinite number of distinct values. Then a.s. there exists a dense (random) set $R \subset [0, \infty)$ such that $N(t) = +\infty$, $\forall t \in R$.

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Idea of the proof.

- Let the statement is not true, then $\exists a < b$ such that

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- $[a, b] = \bigcup_{n=1}^{\infty} \{t \in [a, b] : \ \| \mathrm{pr}_{X_t} \|_{HS}^2 \leq n\}$ w.p.p.

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- $[a, b] = \bigcup_{n=1}^{\infty} \{t \in [a, b] : \ \| \operatorname{pr}_{X_t} \|_{HS}^2 \leq n\}$ w.p.p.
- Since $t \mapsto \| \operatorname{pr}_{X_t} \|_{HS}$ is lower semi-continuous, $\{t : \ \| \operatorname{pr}_{X_t} \|_{HS}^2 \leq n\}$ is closed.

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- Let the statement is not true, then $\exists a < b$ such that

$$N(t) = \| \mathrm{pr}_{X_t} \|_{HS}^2 < \infty, \quad \forall t \in [a, b], \quad \text{w.p.p.}$$

- $[a, b] = \bigcup_{n=1}^{\infty} \{t \in [a, b] : \ \| \mathrm{pr}_{X_t} \|_{HS}^2 \leq n\}$ w.p.p.
- Since $t \mapsto \| \mathrm{pr}_{X_t} \|_{HS}$ is lower semi-continuous, $\{t : \ \| \mathrm{pr}_{X_t} \|_{HS}^2 \leq n\}$ is closed.
- By the Baire category theorem, $(a_1, b_1) \subset [a, b]$, $\exists n \geq 1$ such that

$$N(t) = \| \mathrm{pr}_{X_t} \|_{HS}^2 \leq n, \quad \forall t \in (a_1, b_1), \quad \text{w.p.p.}$$

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Thank you!