On well-posedness and superposition principle for Dean-Kawasaki equation with correlated noise

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Malliavin Calculus and its Applications

joint work with Benjamin Gess and Rishabh Gvalani

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- Usually one approximates f by $f_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} U(\theta, x_k)$, where $x_k \in \mathbb{R}^d$, $k \in \{1, \ldots, n\}$ are parameters which have to be found.
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- We measure the distance between f and f_n by the loss function

$$\mathcal{L}_I[f_n] = \frac{1}{2}\frac{1}{|I|}\sum_{i \in I} |\gamma_i - f_n(\theta_i)|^2 \approx \frac{1}{2}\int_\Theta |f(\theta) - f_n(\theta)|^2 \mathrm{m}(\theta) =: \mathcal{L}[f_n],$$

where m is the distribution of data θ_i.

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  where m is the distribution of data θ_i.
- **Goal:** find parameters x_k, $k \in \{1, \ldots, n\}$, which minimize

$$\mathcal{L}[f_n] = C_f - \frac{1}{n}\sum_{k=1}^{n} F(x_k) + \frac{1}{2n^2}\sum_{k,l=1}^{n} K(x_k, x_l)$$

  for $F(x) = \mathbb{E}_{\mathrm{m}}[f(\theta)U(\theta, x)]$, $K(x, y) = \mathbb{E}_{\mathrm{m}}[U(\theta, x)U(\theta, y)]$.

Stochastic gradient descent

- We can define the parameters using **gradient descent**:

$$\hat{x}_k(t_{i+1}) = \hat{x}_k(t_i) + \left(\nabla F(\hat{x}_k(t_i)) - \frac{1}{n} \sum_{l=1}^{n} \nabla_x K(\hat{x}_k(t_i), \hat{x}_l(t_i)) \right) \Delta t,$$

for $t_i = i\Delta t$, $F(x) = \mathbb{E}_{\mathrm{m}} f(\theta) U(\theta, x)$ and $K(x, y) = \mathbb{E}_{\mathrm{m}} U(\theta, x) U(\theta, y)$.

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- We can define the parameters using **stochastic gradient descent**:

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for $t_i = i\Delta t$, $F_i(t,x) = f(\theta_i)U(\theta_i,x)$ and $K_i(t,x,y) = U(\theta_i,x)U(\theta_i,y)$ and $\{\theta_i, i \in \mathbb{N}\}$ are iid with distribution m.

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- Set $R_i^k(\vec{x}) = \nabla F_i(x_k) - \frac{1}{n} \sum_{l=1}^{n} \nabla_x K_i(x_k, x_l)$. Then

$$\hat{x}_k(t_{i+1}) = \hat{x}_k(t_i) + \mathbb{E}_{\mathrm{m}} R_i^k(\hat{x}(t_i))\Delta t + \sqrt{\Delta t} \left(R_i^k(\hat{x}(t_i)) - \mathbb{E}_{\mathrm{m}} R_i^k(\hat{x}(t_i)) \right) \sqrt{\Delta t},$$

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- This is the Euler-Maruyama scheme for the SDE

$$dx_k(t) = [\nabla F(x_k(t)) - \langle \nabla_x K(x_k(t), \cdot), \mu_t^n\rangle]dt + \sqrt{\Delta t} dB_k(t)$$

$$d[B_k, B_l]_t = \mathrm{Cov}\left(R_i^k, R_i^l\right) dt = \tilde{A}(x_k(t), x_l(t), \mu_t^n)dt, \quad \mu_t^n = \frac{1}{n}\sum_{l=1}^{n} \delta_{x_l(t)}.$$

Equation for empirical measure μ_t^n

$$dx_k(t) = V(x_k(t), \mu_t^n)dt + dB_k(t)$$
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[1] $B : C = \sum_{i,j=1}^{d} B_{i,j} C_{i,j}$, $\langle \psi, \mu_t \rangle = \int_{\mathbb{R}^d} \psi(x)\mu_t(dx)$

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Taking $\varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$, we get for $\mu_t := \mu_t^n$

$$\langle \varphi, \mu_t \rangle = \langle \varphi, \mu_0 \rangle + \frac{1}{2}\int_0^t \left\langle D^2\varphi : A(\cdot, \mu_s), \mu_s \right\rangle ds + \int_0^t \langle \nabla\varphi \cdot V(\cdot, \mu_s), \mu_s \rangle ds + \text{Martingale},$$

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Then $f_n(\theta) = \frac{1}{n}\sum_{k=1}^n U(\theta, x_k(t)) = \int_{\mathbb{R}^d} U(\theta, x)\mu_t(dx)$ should approximate the true function f for large t.[1]

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Main goal

Dean-Kawasaki equation with correlated noise:
For every $\mu_0 = \mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ we are going to find a continuous process μ_t, $t \geq 0$, in $\mathcal{P}_2(\mathbb{R}^d)$ such that $\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$

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Main idea: We will take a cylindrical Wiener process W on $L_2(\Theta, \mathrm{m})$ and assume

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Related works

$$d\mu_t = \frac{1}{2} D^2 : (A(t,\cdot,\mu_t)\mu_t)\, dt - \nabla \cdot (V(t,\cdot,\mu_t)\mu_t)\, dt - \int_\Theta \nabla \cdot (G(t,\cdot,\mu_t,\theta)\mu_t)\, W(d\theta, dt),$$

Well-posedness results for similar SPDEs:

- **Continuity equation in the fluid dynamics and optimal transportation** [Amborsio, Trevisan, Crippa. . .]. There $A = G = 0$.

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The technique from [Kurtz, Xiong] can be applied to our equation if μ_0 has L_2-density!

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Dean-Kawasaki equation with correlated noise

Consider the **Dean-Kawasaki equation with correlated noise**

$$d\mu_t = \frac{1}{2}D^2 : (A(t,\cdot,\mu_t)\mu_t)\,dt - \nabla\cdot(V(t,\cdot,\mu_t)\mu_t)\,dt - \int_\Theta \nabla\cdot(G(t,\cdot,\mu_t,\theta)\mu_t)\,W(d\theta,dt),$$

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in $\mathcal{P}_2(\mathbb{R}^d)$, where

- W is a cylindrical Wiener process in $L_2(\Theta, \mathrm{m})$;
- $V : [0,\infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \Omega \to \mathbb{R}^d$ and $G : [0,\infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \Omega \to L_2(\Theta, \mathrm{m})^d$ satisfy the "standard" measurability assumptions and are bounded on every compact subset of $[0,\infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ a.s.
- $A(t,x,\mu) = \left(\langle G_i(t,x,\mu,\cdot), G_j(t,x,\mu,\cdot)\rangle_{\mathrm{m}}\right)_{i,j\in[d]}$;

Definition of solutions to DK equation

$$d\mu_t = \frac{1}{2} D^2 : (A(t,\cdot,\mu_t)\mu_t)\,dt - \nabla\cdot(V(t,\cdot,\mu_t)\mu_t)\,dt - \int_\Theta \nabla\cdot(G(t,\cdot,\mu_t,\theta)\mu_t)\,W(d\theta,dt),$$

Definition of (weak-strong) solution

A continuous $(\mathcal{F}_t^W)$-adapted process μ_t, $t \geq 0$, in $\mathcal{P}_2(\mathbb{R}^d)$ is a *solution to the DK equation* started from μ_0 if $\forall\ \varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$ a.s. $\forall t \geq 0$

$$\langle \varphi, \mu_t\rangle = \langle \varphi, \mu_0\rangle + \frac{1}{2}\int_0^t \left\langle D^2\varphi : A(s,\cdot,\mu_s), \mu_s \right\rangle ds + \int_0^t \langle \nabla\varphi\cdot V(s,\cdot,\mu_s), \mu_s\rangle\, ds + \int_0^t \int_\Theta \langle \nabla\varphi \cdot G(s,\cdot,\mu_s,\theta), \mu_s\rangle\, W(d\theta, ds)$$

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SDE with interaction

The DK equation with correlated noise has a connection with the SDE with interaction

$$dX(u,t) = V(t, X(u,t), \bar{\mu}_t)dt + \int_\Theta G(t, X(u,t), \bar{\mu}_t, \theta) W(d\theta, dt),$$

$$X(u,0) = u, \quad \bar{\mu}_t = \mu_0 \circ X^{-1}(\cdot, t), \quad u \in \mathbb{R}^d, \quad t \geq 0.$$

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$$X(u,0) = u, \quad \bar{\mu}_t = \mu_0 \circ X^{-1}(\cdot, t), \quad u \in \mathbb{R}^d, \quad t \geq 0.$$

Definition of solution to the SDE with interaction

A family of continuous processes $\{X(u,t),\ t \geq 0\}$, $u \in \mathbb{R}^d$, is a *(strong) solution to SDE with interaction* if $X|_{[0,t]}$ is $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t^W$-measurable, $\bar{\mu}_t = \mu_0 \circ X^{-1}(\cdot, t) \in \mathcal{P}_2(\mathbb{R}^d)$ a.s. $\forall t \geq 0$, and $\forall u \in \mathbb{R}^d$ a.s.

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for all $t \geq 0$.

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A.A. Dorogovtsev Measure-valued processes and stochastic flows, 2007

Well-posedness of SDE with interaction

Lipshitz continuity & linear growths (LC&LG) assumption: $\forall T > 0, \ \exists L > 0$ such that a.s.

$$|V(t,x,\mu) - V(t,y,\nu)| + |||G(t,x,\mu,\cdot) - G(t,y,\nu,\cdot)|||_{\mathrm{m}} \leq L\left(|x-y| + \mathcal{W}_2(\mu,\nu)\right).$$

$\forall t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$
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$$|V(t,0,\delta_0)| + |||G(t,0,\delta_0,\cdot)|||_{\mathrm{m}} \leq L,$$

where δ_0 denotes the δ-measure at 0 on $\mathbb{R}^d$.

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Remark: (LC&LG) implies $|V(t,x,\mu)| + |||G(t,x,\mu,\cdot)|||_{\mathrm{m}} \le L(1 + |x| + \mathcal{W}_2(\mu,\delta_0))$ $\forall t \in [0,T], \ x \in \mathbb{R}^d, \ \mu \in \mathcal{P}_2(\mathbb{R}^d)$.

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Theorem (Dorogovtsev' 07)

Let V, G satisfy (LC&LC). Then for every $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ the SDE with interaction has a unique solution started from μ_0.

Dean-Kawasaki equation and SDE with interaction

SDE with interaction:

$$dX(u,t) = V(t, X(u,t), \bar{\mu}_t)dt + \int_\Theta G(t, X(u,t), \bar{\mu}_t, \theta) W(d\theta, dt),$$
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Lemma

Let X be a solution to the SDE with interaction with $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$.
Then $\bar{\mu}_t = \mu_0 \circ X^{-1}(\cdot, t)$, $t \geq 0$, is a solution to the DK equation.

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Definition: We will say that $\bar{\mu}_t$, $t \geq 0$, is a superposition solution to the DK equation.

Proof of the lemma

$$X(u,t) = u + \int_0^t V(s, X(u,s), \bar{\mu}_s)ds + \int_0^t \int_\Theta G(s, X(u,s), \bar{\mu}_s, \theta)W(d\theta, ds).$$

Proof of the lemma

$$X(u,t) = u + \int_0^t V(s, X(u,s), \bar{\mu}_s) ds + \int_0^t \int_\Theta G(s, X(u,s), \bar{\mu}_s, \theta) W(d\theta, ds).$$

Taking $\varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$ and using Itô's formula, we get

$$\begin{aligned}\varphi(X(u,t)) = \varphi(u) &+ \int_0^t \nabla\varphi(X(u,s)) \cdot V(s, X(u,s), \bar{\mu}_s) ds \\ &+ \frac{1}{2} \int_0^t D^2\varphi(X(u,s)) : A(s, X(u,s), \bar{\mu}_s) ds \\ &+ \int_0^t \int_\Theta \nabla\varphi(X(u,s)) \cdot G(s, X(u,s), \bar{\mu}_s, \theta) W(d\theta, ds), \quad t \geq 0,\end{aligned}$$

where $A(s,x,\mu) = \| G(s,x,\mu,\cdot)\|_{\mathrm{m}}^2$.

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Then we integrate the LHS and RHS with respect to $\mu_0(du)$ and use $\int_{\mathbb{R}^d} \psi(X(u,t))\mu_0(du) = \langle \psi, \bar{\mu}_t \rangle$.

Uniqueness and superposition principle

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Corollary

Let V, G satisfy (LC&LG). Then the DK equation has a unique solution iff it has **only** superposition solutions.

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Transformation ψ_t

$$\psi_t(x) = x - \sigma w_t, \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

$\psi_t = (\psi_t^1, \dots, \psi_t^d)$ is a solution to the stochastic transport equation:

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and its inverse $\phi_t(x) = \psi_t^{-1}(x) = x + \sigma w_t$ solves

$$\phi_t(x) = x + \int_0^t \sigma dw_t$$

Come back to DK equation

Let μ_t, $t \geq 0$, satisfies $\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$

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We introduce the field of martingales

$$M(x,t) = \int_0^t g(s,x,\theta) W(d\theta, ds), \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

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$$+ \int_0^t \langle \nabla \varphi \cdot V(s, \cdot, \mu_s), \mu_s \rangle \, ds + \int_0^t \int_\Theta \langle \nabla \varphi \cdot G(s, \cdot, \mu_s, \theta), \mu_s \rangle \, W(d\theta, ds)$$

We want to show that $\mu_t = \mu_0 \circ X^{-1}(\cdot, t)$, $t \geq 0$.

We freeze μ_t in the coefficients: $a(t,x) = A(t,x,\mu_t)$, $v(t,x) = V(t,x,\mu_t)$, $g(t,x,\theta) = G(t,x,\mu_t,\theta)$ and $\tilde{a}(t,x,y) = \tilde{A}(t,x,y,\mu_t) = \langle G(t,x,\mu_t,\theta), G(t,y,\mu_t,\theta) \rangle_{\mathrm{m}}$.

We introduce the field of martingales

$$M(x,t) = \int_0^t g(s,x,\theta) W(d\theta, ds), \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

and consider a solution $\psi_t(x) = (\psi_t^1(x), \dots, \psi_t^d(x))$ to

$$\psi_t^k(x) = x^k - \int_0^t \nabla \psi_s^k(x) \cdot M(x, \circ ds), \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad k \in \{1, \dots, d\}$$

Come back to DK equation

Let μ_t, $t \geq 0$, satisfies $\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$

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$$= x^k - \int_0^t \int_\Theta \nabla\psi_s^k(x) \cdot g(s,x,\theta) W(d\theta, ds)$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \nabla_x \cdot \left(\tilde{a}(s,x,y) \cdot \nabla\psi_s^k(x) \right) \delta_x(dy) ds$$

Assumptions on coefficients

We introduce the norms

$$\|f\|_{m+\delta} = \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{1+|x|} + \sum_{1\le|\alpha|\le m} \sup_{x\in\mathbb{R}^d} |D^\alpha f(x)| + \sum_{|\alpha|=m} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\delta},$$

$$\|h\|^\sim_{m+\delta} = \sup_{x,y\in\mathbb{R}^d} \frac{|h(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\le|\alpha|\le m} \sup_{x,y\in\mathbb{R}^d} |D^\alpha_x D^\alpha_y h(x,y)|$$
$$+ \sum_{|\alpha|=m} \sup_{x\neq x', y\neq y'} \frac{|D^\alpha_x D^\alpha_y h(x,y) - D^\alpha_{x'} D^\alpha_y h(x',y) - D^\alpha_x D^\alpha_{y'} h(x,y') + D^\alpha_{x'} D^\alpha_{y'} h(x',y')|}{|x-x'|^\delta |y-y'|^\delta}.$$

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$$+ \sum_{|\alpha|=m} \sup_{x\neq x',y\neq y'} \frac{|D_x^\alpha D_y^\alpha h(x,y) - D_{x'}^\alpha D_y^\alpha h(x',y) - D_x^\alpha D_{y'}^\alpha h(x,y') + D_{x'}^\alpha D_{y'}^\alpha h(x',y')|}{|x-x'|^\delta |y-y'|^\delta}.$$

Smoothness of coefficients (SM):

$$\sup_{t\in[0,T],\mu\in\mathcal{P}_2(\mathbb{R}^d)} \left(\|V^i(t,\cdot,\mu)\|_{1+\delta} + \|\tilde{A}^{i,j}(t,\cdot,\mu)\|^{\sim}_{3+\delta} \right) < \infty, \quad i,j \in \{1,\dots,d\}$$

Uniqueness of solutions to DK equation

Theorem

Let the coefficients to the DK equation

$$d\mu_t = \frac{1}{2} D^2 : (A(t, \cdot, \mu_t)\mu_t)\, dt - \nabla \cdot (V(t, \cdot, \mu_t)\mu_t)\, dt - \int_\Theta \nabla \cdot (G(t, \cdot, \mu_t, \theta)\mu_t)\, W(d\theta, dt),$$

satisfy assumptions (LC&LG) and (SM). Then for every $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a unique solution started from μ_0 which also is a superposition solution.

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satisfy assumptions (LC&LG) and (SM). Then for every $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a unique solution started from μ_0 which also is a superposition solution.

The idea of the proof is to remove the noise and the second order term from the equation by the transformation

$$\rho_t = \mu_t \circ \psi_t^{-1},$$

where $\psi_t^k(x) = x^k - \int_0^t \nabla\psi_s^k(x)\cdot M(x, \circ ds)$ and $M(x,t) = \int_0^t g(s,x,\theta)W(d\theta, ds)$.

Well-posedness of the stochastic transport equation

Lemma

Let the coefficients to the DK equation satisfy **(SM)** assumption.
Then $\forall k \in \{1, \dots, d\}$ $\exists \delta' \in (0, \delta)$ and an $(\mathcal{F}_t^W)$-adapted continuous $\mathcal{C}^{3,\delta'}$-valued process ψ_t^k, $t \geq 0$, satisfying a.s.

$$\psi_t^k(x) = x^k - \int_0^t \nabla \psi_s^k(x) \cdot M(x, \circ ds), \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

Moreover, a.s. $\forall t \geq 0$ the map $\psi_t = (\psi_t^1, \dots, \psi^d) : \mathbb{R}^d \to \mathbb{R}^d$ is invertible and $\phi_t := \psi_t^{-1}$ is an $(\mathcal{F}_t^W)$-adapted continuous $\mathcal{C}^{3,\delta'}(\mathbb{R}^d)$-valued stochastic process that satisfies the equation

$$\phi_t(x) = x + \int_0^t M(\phi_s(x), \circ ds), \quad t \geq 0, \quad x \in \mathbb{R}^d$$

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$$= x + \int_0^t \int_\Theta g(s, \phi(s,x), \theta) W(d\theta, ds) + \frac{1}{2} \int_0^t (\nabla_x \cdot \tilde{a})(s, \phi(s,x), \phi(s,x)) ds$$

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For the proof see in [Kunita Stochastic flows and SDEs]

Transformation of space

For the solution μ_t, $t \geq 0$, we consider

$$\rho_t = \mu_t \circ \psi_t^{-1}, \quad t \geq 0$$

Proposition

Let the coefficients of the DK equation satisfy **(SM)** assumption. Then ρ_t, $t \geq 0$, is a solution to the transport equation

$$d\rho_t = -\nabla(b(t,\cdot)\rho_t)dt, \quad \rho_0 = \mu_0,$$

i.e. $\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}^d)$ a.s.

$$\langle \rho_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \nabla\varphi \cdot b(s,\cdot), \rho_s \rangle \, ds, \quad t \geq 0,$$

where $b(t,x) = \tilde{b}(t, \psi_t^{-1}(x))$ and

$$\tilde{b}^k(t,x) = \nabla\psi_t^k(x) \cdot v(t,x) - \frac{1}{2}\int_{\mathbb{R}^d} \nabla_x \cdot \left(\tilde{a}(t,x,y) \cdot \nabla\psi_t^k(x)\right) \delta_x(dy).$$

Idea of proof of the proposition

$$\langle \varphi, \mu_t \rangle = \text{bdd var. proc.} + \int_0^t \int_\Theta \langle \nabla \varphi \cdot g(s, \cdot, \theta), \mu_s \rangle \, W(d\theta, ds)$$

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Note that

$$\varphi \circ \psi_t(x) = -\sum_{k=1}^d \int_0^t \int_\Theta \partial_k \varphi \nabla\psi_s^k(x) \cdot g(s,x,\theta) W(d\theta, ds) + \text{bdd var. proc.}$$

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$$- \int_0^t \int_\Theta \langle \nabla (\varphi \circ \psi_t) \cdot g(s, \cdot, \theta), \mu_s \rangle W(d\theta, ds) + \text{bdd var. proc.}$$

Proof of the uniqueness result

- We know that $\rho_t = \mu_t \circ \psi_t^{-1}$ is a solution to the transport equation with random coefficient:

$$d\rho_t = -\nabla(b(t,\cdot)\rho_t)dt, \quad \rho_0 = \mu_0.$$

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- Using the duality argument, we can prove the uniqueness to the equation above.
- The direct computation shows that $\bar{\rho}_t = \mu_0 \circ Y^{-1}(\cdot, t)$ is the solution to the transport equation, where $Y(u,t) = \psi_t \circ X(u,t)$ for X being a unique solution to
$$dX(u,t) = v(t, X(u,t))dt + \int_\Theta g(t, X(u,t), \theta)W(d\theta, dt),$$
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- Then $\mu_t \circ \psi_t^{-1} = \rho_t = \bar{\rho}_t = \mu_0 \circ Y^{-1}(\cdot, t) = \mu_0 \circ X^{-1}(\cdot, t) \circ \psi_t^{-1}$.

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- Hence $\mu_t = \mu_0 \circ X^{-1}(\cdot, t)$.

Thank you!