Sticky-reflected stochastic heat equation driven by colored noise

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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

UNIVERSITÄT LEIPZIG

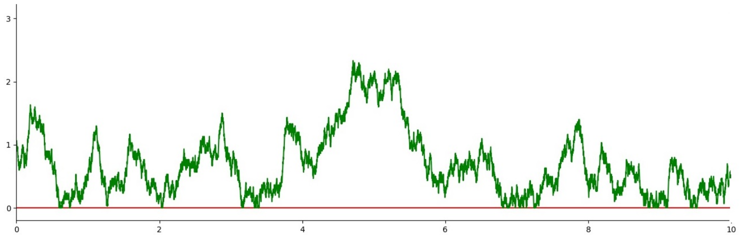
NATIONAL ACADEMY OF SCIENCES OF UKRAINE
INSTITUTE OF MATHEMATICS

Introduction of the equation

Sticky-reflected Brownian motion

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} dw(t),$$
$$x(0) = x_0 \geq 0,$$

where $\lambda > 0$ and w is an 1-dim Brownian motion.



The equation admits only a **weak solution** which is **unique in law** (Engelbert and Peskir, 2014)

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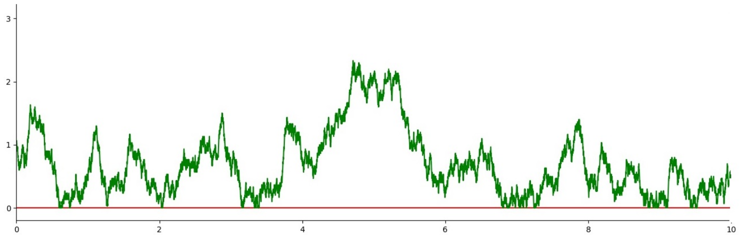

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Stochastic heat equation

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + \dot{W}_t, \quad t > 0, \quad u \in (0,1),$$
$$X_0 = g, \quad X_t(0) = X_t(1) = 0,$$

where $\dot{W}$ is a space-time white noise and $g \in C[0,1]$.

Definition of weak solution

A continuous process $X : [0,\infty) \times [0,1] \to \mathbb{R}$ is called a **weak solution** to the SHE if for any $\varphi \in C^2[0,1]$ with $\varphi(0) = \varphi(1) = 0$

$$M_t^\varphi := \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \frac{1}{2}\int_0^t \langle X_s, \varphi'' \rangle \, ds$$

is a martingale with quadratic variation

$$[M^\varphi]_t = \int_0^t \|\varphi\|_{L_2}^2 ds,$$

where $\langle X_t, \varphi \rangle = \int_0^1 X_t(u)\varphi(u)du$. (Well-posedness – Funaki, 1983)

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Sticky-reflected stochastic heat equation

Stochastic heat equation on $[0,1]$

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Reflected SHE (D. Nulart and É. Pardoux '92)

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + L_t + \dot{W}_t$$

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There exists a unique continuous process $X : [0,1] \times [0,\infty) \to \mathbb{R}$ and a measure (local time) L on $[0,1] \times [0,\infty)$ satisfying the reflected SHE. (D. Nulart and É. Pardoux '92)

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Formulation of the main result

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where Q is non-negative definite self-adjoint Hilbert-Schmidt operator in $L_2[0,1]$

Solution to sticky-reflected SHE

A continuous process $X : [0,\infty) \times [0,1] \to \mathbb{R}$ is called a **weak solution** to the sticky-reflected SHE if for any $\varphi \in C^2[0,1]$ with $\varphi(0) = \varphi(1) = 0$

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Theorem K. 2021

If $\chi^2 > 0$ a.e., then the sticky-reflected SHE admits a weak solution.

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Meaning of assumtion $\chi^2 > 0$

The equation

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} dw(t)$$
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has no solution

$\chi^2 = \sum_{k=1}^{\infty} \mu_k^2 e_k^2 > 0$ means that the solution X to the equation

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feels a noise at any point of $[0, 1]$.

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Description of the idea of construction of solution

using the equation

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} dw(t)$$

Step I. Approximation sequence

Consider the SDE for sticky-reflected BM:

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} dw(t),$$
$$x(0) = x_0 \geq 0.$$

We approximate its solution by the solutions to the SDE

$$dx_n(t) = \lambda(1 - \kappa_n^2(x_n(t)))dt + \kappa_n(x_n(t))dw(t),$$
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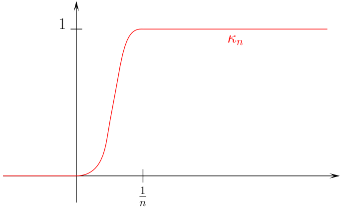
which have non-negative strong solutons $x_n(t) \geq 0$.



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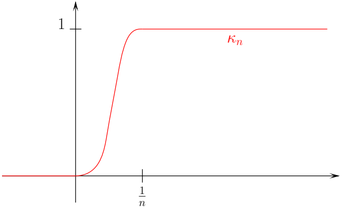

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Problem of approximation

Once can show that $\{x_n,\ n \geq 1\}$ is tight in $C[0,\infty)$ $\Longrightarrow$

$$x_n \to x \quad \text{in} \quad C[0,\infty)$$

along a subsequence.

But

$$\begin{array}{ccccc} x_n(t) = x_0 & + \int_0^t \lambda(1-\kappa_n^2(x_n(s)))ds & + \int_0^t \kappa_n(x_n(s))dw(s), \\ \downarrow & \not\downarrow & \not\downarrow \\ x(t) = x_0 & + \int_0^t \lambda \mathbb{I}_{\{x(s)=0\}} ds & + \int_0^t \mathbb{I}_{\{x(s)>0\}} dw(s) \end{array}$$

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Step II. Convergence in an appropriate space

$$x_n(t) = x_0 + \int_0^t \lambda(1 - \kappa_n^2(x_n(s)))ds + \int_0^t \kappa_n(x_n(s))dw(s),$$

Using tightntess argument, one has

$$\begin{aligned}
x_n(t) &\to x(t) \\
a_n(t) := \int_0^t \lambda(1 - \kappa_n^2(x_n(s)))ds &\to a(t) \\
\eta_n(t) := \int_0^t \kappa_n(x_n(s))dw(s) &\to \eta(t) \\
[\eta_n]_t = \int_0^t \kappa_n^2(x_n(s))ds &\to \rho(t)
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in $C[0, \infty)$ in distribution along a subsequence.

Step III. Properties of the limit process

$$x_n(t) \to x(t), \quad a_n(t) := \int_0^t \lambda(1-\kappa_n^2(x_n(s)))ds \to a(t)$$

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We remark that

- $x(t) = x_0 + a(t) + \eta(t) \geq 0$
- η is a continuous martingale
- $[\eta(t)]_t = \rho(t)$
- $\kappa_n^2(x_n)$ is tight in the weak topology of $L_2[0,T]$, therefore,

$$\kappa_n^2(x_n) \to \dot{\rho} \in L_2[0,T] \quad \text{and} \quad \rho(t) = \int_0^t \dot{\rho}(s)ds$$

- $a(t) = \lambda t - \lambda\rho(t) = \int_0^t \lambda(1-\dot{\rho}(s))ds$

We need to show that $\dot{\rho}(s) = \mathbb{I}_{\{x(s)>0\}}$!

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- $a(t) = \lambda t - \lambda\rho(t) = \int_0^t \lambda(1 - \dot{\rho}(s))ds = \int_0^t \lambda(1 - \mathbb{I}_{\{x(t)>0\}})ds = \int_0^t \lambda \mathbb{I}_{\{x(s)=0\}} ds$

We need to show that $\dot{\rho}(s) = \mathbb{I}_{\{x(s)>0\}}$!

Key observation

Lemma

If x is a continuous non-negative semimartingale with q.v.

$$[x]_t = \int_0^t \sigma^2(s)ds,$$

then

$$\sigma^2(s) = \sigma^2(s)\mathbb{I}_{\{x(s)>0\}} \quad s\text{-a.e.}$$

Proof.

$$\int_0^t \sigma^2(s)\mathbb{I}_{\{x(s)=0\}}ds = \int_0^t \mathbb{I}_{\{0\}}(x(s))d[x]_s$$
$$= \int_{-\infty}^{+\infty} \mathbb{I}_{\{0\}}(x)L_t^x dx = 0, \quad t \geq 0$$

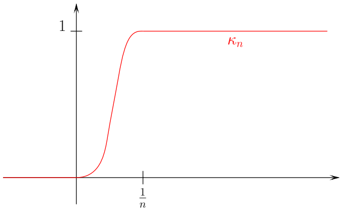
whre L_t^x is the local time of x.

Step IV. Identification of quadratic variation

Remind

- $x(t) = x_0 + a(t) + \eta(t) \geq 0$ is a continuous semimartingale
- $[x]_t = [\eta]_t = \int_0^t \dot\rho(s)ds$
- $\kappa_n^2(x_n) \to \dot\rho$ in a weak topology of $L_2[0,1]$ along a subsequence

Since $\dot\rho = \dot\rho \mathbb{I}_{\{x(s)>0\}}$ and



$$\kappa_n^2(y_n)\mathbb{I}_{\{y>0\}} \to \mathbb{I}_{\{y>0\}},$$

as $y_n \to y$, we get

$$\int_0^t \dot\rho(s)dt = \int_0^t \dot\rho \mathbb{I}_{\{x(s)>0\}}ds = \lim_n \int_0^t \kappa_n^2(x_n(s))\mathbb{I}_{\{x(s)>0\}}ds$$

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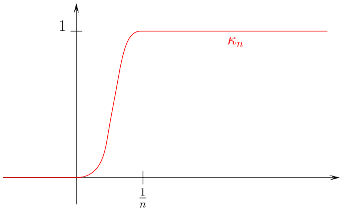

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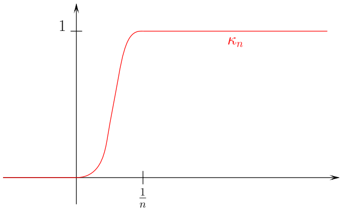

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Proof of existence of solution to

sticky-reflected SHE

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + \lambda \mathbb{I}_{\{X_t=0\}} + \mathbb{I}_{\{X_t>0\}} Q\dot{W}_t$$

Discrete equation

We discretize only the space variable $u \in [0,1]$ by $\frac{k}{n}$, $k = 1, \dots, n$.

Set $\pi_k^n = \mathbb{I}_{[\frac{k-1}{n}, \frac{k}{n})}$ and define

$$w_k(t) := \sqrt{n} \int_0^t \int_0^1 (Q\pi_k^n)(u) W(du, ds)$$

Consider the following SDE

$$dx_k(t) = \frac{1}{2}\Delta^n x_k(t)dt + \mathbb{I}_{\{x_k(t)=0\}}dt + \sqrt{n}\mathbb{I}_{\{x_k(t)>0\}}dw_k(t), \quad k = 1, \dots, n,$$

with $x_0(t) = x_{n+1}(t) = 0$ and $\Delta^n x_k = n^2 (x_{k+1} + x_{k-1} - 2x_k)$

Set

$$X_t^n(u) = x_k(t), \quad \frac{k-1}{n} \le u < \frac{k}{n}, \quad u \in [0,1].$$

Remark that $X^n \ge 0$.

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Convergence result

For every $\varphi \in \mathcal{C}^2[0,1]$ with $\varphi(0) = \varphi(1) = 0$,

$$\langle X_t^n, \varphi \rangle = \langle X_0^n, \varphi \rangle + \frac{1}{2} \int_0^t \left\langle X_s^n, \tilde{\Delta}^n \varphi \right\rangle ds + \int_0^t \left\langle \mathbb{I}_{\{X_s^n = 0\}}, \varphi \right\rangle ds + B_\varphi^n$$

where B_φ^n is a continuous martingale with $[B_\varphi]_t = \left\| Q(\mathbb{I}_{\{X_t^n > 0\}} \varphi) \right\|^2$.

There exists a subsequence n_k, $k \geq 1$, and a continuous process X such that

- $X^{n_k}(u,t) \to X(u,t)$, $\forall u, t$;
- $\tilde{\Delta}^{n_k} \varphi \to \varphi''$;
- $\mathbb{I}_{\{X_t^{n_k} > 0\}} \to \sigma_t$;
- $\mathbb{I}_{\{X_t^{n_k} = 0\}} = 1 - \mathbb{I}_{\{X_t^{n_k} > 0\}} \to 1 - \sigma_t$.

Hence

$$\langle X_t, \varphi \rangle = \langle g, \varphi \rangle + \frac{1}{2} \int_0^t \langle X_s, \varphi'' \rangle ds + \int_0^t \langle (1 - \sigma_s), \varphi \rangle ds + B_\varphi$$

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Equivalently

$$\frac{\partial X_t}{\partial t} = \frac{1}{2} \frac{\partial^2 X_t}{\partial u^2} + (1 - \sigma) + \sigma Q \dot{W},$$

$$X_t(0) = X_t(1) = 0, \quad X_0(u) = g(u).$$

Identification of coefficient σ

Proposition (K., 2020)

Let X solves the equation

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and $X \geq 0$. Then $\sigma = \mathbb{I}_{\{X_t>0\}}\sigma$.

Identification of the coefficients

By the previous proposition,

$$\sigma_t(u) = \mathbb{I}_{\{X_t(u)>0\}}\sigma_t(u) = \lim_{k\to\infty} \mathbb{I}_{\{X_t(u)>0\}}\mathbb{I}_{\left\{X_t^{n_k}(u)>0\right\}} = \mathbb{I}_{\{X_t(u)>0\}}$$

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Idea of proof of the key proposition

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Proof

Analog of Ito's formula applid to F_ε:

$$\langle F_\varepsilon(X_t) - F_\varepsilon(X_0), 1\rangle = -\frac{1}{2}\int_0^t \left\langle F_\varepsilon''(X_s)\dot{X}_s, \dot{X}_s \right\rangle ds + \int_0^t \langle F_\varepsilon'(X_s), a_s\rangle\, ds + \frac{1}{2}\int_0^t \langle Q[\sigma F_\varepsilon''(X_s)\cdot], Q[\sigma\cdot]\rangle_{HS}\, ds + M_{F_\varepsilon}(t),$$

where

$$F_\varepsilon(x) := \int_{-\infty}^{x}\int_{-\infty}^{y} \psi_\varepsilon(r)\,dy\,dr,$$

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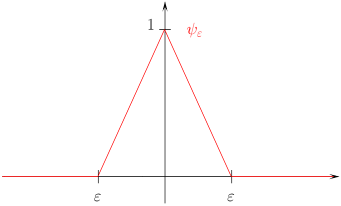

Hence all green terms $\to 0$ and red term $\to \int_0^t \langle Q[\sigma \mathbb{I}_{\{X_s=0\}}\cdot], Q[\sigma\cdot]\rangle_{HS}\, ds$

$\implies$ We can replace σ by $\mathbb{I}_{\{X_s>0\}}\sigma$

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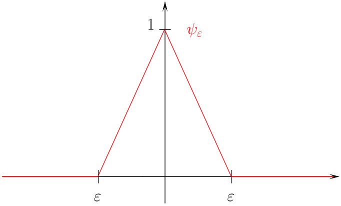

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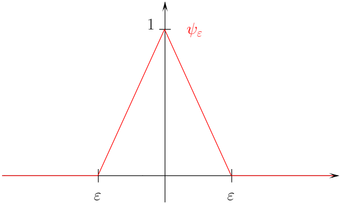

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Open problem and references

Open problems:

- Is a solution to the equation unique?
- Does the solution of the equation with the identity operator Q exists?
- What is the invariant measure for the dynamics?
- How much time does the equation spend at zero?

Vitalii Konarovskyi,
Sticky-Reflected Stochastic Heat Equation Driven by Colored Noise
Ukrain. Math. J., Vol. 72, no. 9, 2021
(arXiv:2005.11773)

Vitalii Konarovskyi,
Coalescing-Fragmentating Wasserstein Dynamics: particle approach
(arXiv:1711.03011)

Thank you for your attention!