A Particle Model for Wasserstein Type Diffusion

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Bielefeld Stochastic Afternoon

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Dean-Kawasaki Equation

The Dean-Kawasaki equation for non-interacting particle systems:

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla \cdot \left(\sqrt{\mu_t}\dot{W}_t\right)$$

It is used e.g. for description of particle density in the Langevin dynamics.
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Definition of (martingale) solution

A continuous process $\mu_t \in \mathcal{P}(\mathbb{R}^d)$, $t \geq 0$, is a solution to the Dean-Kawasaki equation if, for every $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$

$$M_\varphi(t) = \langle \varphi, \mu_t \rangle - \langle \varphi, \mu_0 \rangle - \frac{\alpha}{2}\int_0^t \langle \Delta\varphi, \mu_s \rangle ds$$

is a martingale with quadratic variation

$$\int_0^t \langle |\nabla \varphi|^2, \mu_s \rangle ds.$$

Particle solutions to DK equation

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By Ito's formula,

$$\langle \varphi, \mu_t \rangle = \frac{1}{n} \sum_{k=1}^{n} \varphi(B_k(t)) = \langle \varphi, \mu_0 \rangle + \frac{1}{2n} \sum_{k=1}^{n} \int_0^t \Delta\varphi(B_k(s)) d(ns)$$
$$+ \frac{1}{n} \sum_{k=1}^{n} \int_0^t \nabla\varphi(B_k(s)) \cdot dB_k(s)$$

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Hence

$$M_\varphi(t) = \langle \varphi, \mu_t \rangle - \langle \varphi, \mu_0 \rangle - \frac{n}{2} \int_0^t \langle \Delta\varphi, \mu_s \rangle \, ds$$

is a martingale with q.v. $[M_\varphi]_t = \int_0^t \langle |\nabla\varphi|^2, \mu_s \rangle \, ds$.

DK equation: ill-posedness vs. triviality

Therefore,

$$\mu_t = \frac{1}{n}\sum_{k=1}^{n} \delta_{B_k(t)}, \quad t \geq 0, \tag{1}$$

is a solution to the DK equation

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

with $\alpha = n$.

Theorem (K./ Lehmann/ Renesse/ '19)

Only for $\alpha = n \in \mathbb{N}$ and $\mu_0 = \frac{1}{n}\sum_{k=1}^{n}\delta_{x_k}$ the DK equation has a solution. Moreover, it is unique and defined by (1).

Proof of the theorem: basic properties of solutions

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

- **The equation preserves the total mass**, i.e $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$.
  Take $\varphi \equiv 1$. Then

$$\mu_t(\mathbb{R}^d) = \langle\varphi, \mu_t\rangle = \langle\varphi, \mu_0\rangle + \int_0^t \langle\Delta\varphi, \mu_s\rangle ds + M_\varphi(t)$$

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  where the q.v. $[M_\varphi]_t = \int_0^t \langle|\nabla\varphi|^2,\mu_s\rangle ds = 0$.
- **Laplace duality:**
  $$\mathbb{E}e^{-\langle f,\mu_t\rangle} = e^{-\langle v(t),\mu_0\rangle},$$
  where v is a solution to the Hamilton-Jacobi equation:
  $$\begin{cases}\frac{\partial v}{\partial t} = \frac{\alpha}{2}\Delta v - \frac{1}{2}|\nabla v|^2, \\ v\big|_{t=0} = f\end{cases}$$

$$d_s e^{-\langle v(t-s),\mu_s\rangle} = e^{-\langle v(t-s),\mu_s\rangle} \cdot \left[\langle -\partial_s v(t-s) - \frac{\alpha}{2}\Delta v(t-s) + \frac{1}{2}|\nabla v(t-s)|^2, \mu_s\rangle\right] ds + dM$$

Proof of the theorem: generating function of $\mu_t(A)$

H-J equation:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\alpha}{2}\Delta v - \frac{1}{2}|\nabla v|^2, \\ v\big|_{t=0} = f \end{cases}$$

Solution to H-J equation: $V_t f = -\alpha \ln\left(P_t e^{-\frac{1}{\alpha}f}\right)$,

where $u(t) = P_t g$ is the solution to the heat equation:

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Lemma.

For $A \subset \mathbb{R}^d$ and $t \geq 0$, one has

$$\mathbb{E}s^{\alpha\mu_t(A)} = e^{\alpha\langle\mu_0, \ln(1+(s-1)P_t\mathbb{I}_A)\rangle}, \quad s > 0.$$

$$\mathbb{E}e^{-r\alpha\mu_t(A)} = \mathbb{E}e^{-\langle\mu_t, r\alpha\mathbb{I}_A\rangle} = e^{-\langle\mu_0, V_t(r\alpha\mathbb{I}_A)\rangle}$$

$$= e^{-\left\langle\mu_0, -\alpha\ln\left(P_t e^{-r\mathbb{I}_A}\right)\right\rangle} = e^{\alpha\langle\mu_0, \ln\left(1+(e^{-r}-1)P_t\mathbb{I}_A\right)\rangle}, \quad r > 0$$

Proof of the theorem: conclusion

- $\mathbb{E}s^{\alpha\mu_t(A)} = e^{\alpha\langle\mu_0,\ln(1+(s-1)P_t\mathbb{I}_A)\rangle}$ $\quad t \geq 0, \quad A \subset \mathbb{R}^d$;

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Lemma.

Let ξ be a nonnegative random variable on $\mathbb{R}$ and $\forall n \geq 1$

$$\mathbb{E}s^{\xi} = \sum_{k=0}^{n} s^k p_k + o(s^n), \quad s \to 0+ .$$

Then $\xi \in \mathbb{N} \cup \{0\}$ a.s. and $\mathbb{P}\{\xi = k\} = p_k, \ k \geq 0$.

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- Making $A \uparrow \mathbb{R}$, $\alpha\mu_t(A) \to \alpha \in \mathbb{N}$;
- Making $t \to 0+$, we get $\mu_0 = \frac{1}{\alpha}\sum_{i=1}^{\alpha} \delta_{x_i}$.

DK equation for interacting particle systems

Dean-Kawasaki equation for interacting particle system:

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right) + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

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Girsanov's transformation $d\tilde{\mathbb{P}} = e^{M^F(t)-\frac{1}{2}[M^F]_t}d\mathbb{P}$, where M^F is the martingale part of $F(\mu_t)$, $t \geq 0$, gives that μ_t is a solution to

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Theorem (K./ Lehmann/ Renesse '20)

Let $F \in \mathcal{C}_b^2(\mathcal{M}_F(\mathbb{R}^d))$. Then the Dean-Kawasaki equation has a solution only for $\alpha = n \in \mathbb{N}$ and $\mu_0 = \frac{1}{n}\sum_{k=1}^n \delta_{x_k}$. Moreover, it is uniquely defined by

$$\mu_t = \frac{1}{n}\sum_{k=1}^n \delta_{X_k(t)}, \quad t \geq 0.$$

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Corrected Dean-Kawasaki equations

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla \cdot \left(\sqrt{\mu_t}\dot{W}_t\right)$$

- **Correction of diffusion:**

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla \cdot \left(\sigma(\mu_t)\dot{W}_t^c\right)$$

J. Zimmer, F. Cornalba, T. Shardlow, B. Gess, B. Fehrman, M. Mariani. . .

  - Well-posedness;
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  - Particle approximation, etc.

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- **Correction of "drift"**

$$\frac{\partial}{\partial t}\mu_t = \Gamma(\mu_t) + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

  M. von Renesse, S. Andres, L. Dello Schiavo, V. Marx...

  - Connection with geometry of the Wasserstein space;
  - Assymptotic behaviour;
  - Particle approximation, etc.

Main goal

Goal: propose a system of interacting diffusion particles on $\mathbb{R}$ with masses such that the associated measure-valued process is a weak solution to a corrected Dean-Kawasaki equation

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Why the model can be interesting:

- The interesting particle system is a physical improvement of already existing models.

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- The model satisfies the Varadhan formula for short times which is governed by the quadratic Wasserstein distance.
- In reversible case, it has a new invariant measure on the space of probability measures on $\mathbb{R}$ with full support.
- It is a (non-unique) particle solution to a corrected Dean-Kawasaki equation on $\mathcal{P}(\mathbb{R})$.

Key observation

Let X_1 and X_2 be independent continuous semimartingales with quadratic variation

$$[X_k]_t = a_k t$$

Consider

$$\mu_t = m_1 \delta_{X_1(t)} + m_2 \delta_{X_2(t)}, \quad t \geq 0$$

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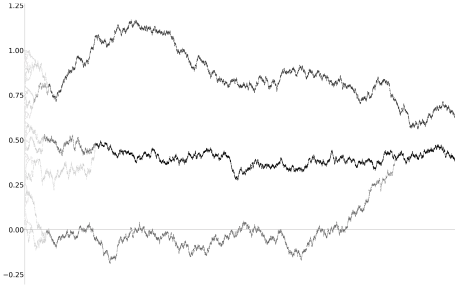
$$a_k = \frac{1}{m_k}$$

$\Longrightarrow$ We can not construct a solution to the DK equation with some Γ started from the Lebesgue measure on $[0, 1]$, where massive particles move as independent, e.g., Brownian motions.

A coalescing particle system

Modified massive Arratia flow on $\mathbb{R}$

- Brownian particles start from points with masses;
- they move independently and coalesce after meeting;
- particles sum their masses after meeting and diffusion rate is inversely proportional to the mass.



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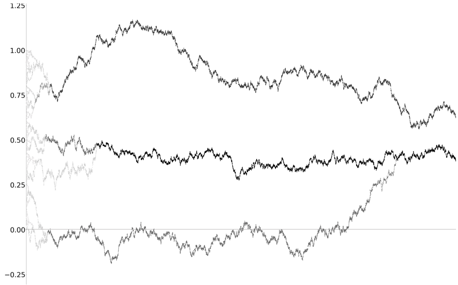
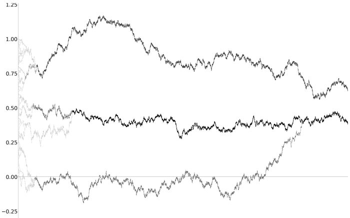


The model is a physical improvement of the Arratia flow, where particles do not change their diffusion rates.

(Arratia '79; Le Jan, Raimond '04; Schertzer, Sun, Swart '14; Berestycki, Garban, Sen '15)

Mathematical description of MMAF



Theorem (K., '17)

There exists a family of continuous processes $X(u,t),\ t \geq 0$, $u \in [0,1]$ such that

1. $X(u,0) = u$, $u \in [0,1]$;
2. $X(u,\cdot)$ is a continuous martingale;
3. $X(u,t) \leq X(v,t)$, $u < v$;
4. $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$, $m(u,s) = \mathrm{Leb}\{w : X(w,t) = X(u,t)\}$.

$X(u,t)$ is the position of particle at time t started from $u \in [0,1]$

MMAF as a solution to corrected DK equation

Theorem **(K./ Renesse '19)**

- The evolution of particle masses $\mu_t = X(\cdot, t)_{\#}$ Leb satisfies the equation

$$\frac{\partial}{\partial t}\mu_t = \frac{1}{2}\Delta\mu_t^* + \nabla \cdot (\sqrt{\mu_t}\dot{W}_t),$$

with $\mu_t^* = \sum_{x \in \mathrm{supp}\,\mu_t} \delta_x$ and $\mu_0 = \mathrm{Leb}\,\big|_{[0,1]}$.

Short-time asymptotic of a Brownian motion

Short-time asymptotic formula for a heat kernel

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{2t}} \sim e^{-\frac{\|x-y\|^2}{2t}}, \quad t \to 0+ .$$

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Generalizations

- Heat equation with variable coefficients in $\mathbb{R}^n$ (Varadhan (CPAM '67))
- Smooth Riemannian manifold with Ricci curvature bound (P. Li and S.-T. Yau (Acta Math. '86))
- Lipschitz Riemannian manifold without any sort of curvature bounds (J. Norris (Acta Math. 97))
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Corollary

If B_t, $t \geq 0$, is a Brownian motion on a Riemannian manifold, then

$$\mathbb{P}_x\{B_t = y\} \sim e^{-\frac{d^2(x,y)}{2t}}, \quad t \to 0+,$$

with d being the Riemannian distance.

Connection with optimal transport

Theorem (K./ Renesse '19)

The process $\mu_t = Y(\cdot, t)\big|_{\#}\mathrm{Leb}$, $t \geq 0$, which describes the evolution of particle masses in the modified massive Arratia flow satisfies Varadhan's formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0,\nu)}{2t}}, \quad t \to 0+,$$

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Quadratic Wasserstein distance: $d_{\mathcal{W}}(\nu_1, \nu_2) = \inf_{\xi_1 \sim \nu_1, \xi_2 \sim \nu_2} \left(\mathbb{E}|\xi_1 - \xi_2|^2\right)^{\frac{1}{2}}$

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$(\mathcal{P}_2(\mathbb{R}), d_{\mathcal{W}})$ has an inf.-dim. Riemannian structure (F. Otto (JFA, '01)).

Idea of proof of Varadhan's formula

Idea of proof of $\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0,\nu)}{2t}}, \quad t \to 0+$**:**

- Since $X(u,t) \le X(v,t)$ for $u < v$, one can show that $X(\cdot, t)$, $t \ge 0$, is a continuous process in $L_2^{\uparrow} \subset L_2[0,1]$.

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- $\{X(\cdot,\varepsilon t),\ t \in [0,T]\}_{\varepsilon>0}$ satisfies the LDP in $\mathcal{C}([0,T], L_2^{\uparrow})$ with rate function

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}(t)\|^2_{L_2} dt, & \varphi \in H^2_{\mathrm{id}}([0,T], L_2^{\uparrow}), \\ +\infty, & \text{otherwise.} \end{cases}$$

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$$I(g) = \frac{1}{2}\|\mathrm{id} - g\|^2_{L_2}, \quad g \in L_2^{\uparrow}.$$

- We use the isometry $\|g - f\|_{L_2} = d_{\mathcal{W}}(\nu_g, \nu_f)$, where $\nu_g = g_{\#} \operatorname{Leb}|_{[0,1]}$ and $\nu_f = f_{\#} \operatorname{Leb}|_{[0,1]}$, $g, f \in L_2^{\uparrow}$.

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Sticky-reflected interaction

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1. $X(u,0) = u$, $u \in [0,1]$
2. $X(u,\cdot)$ is a continuous martingale
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Sticky-reflected interaction

Can we replace the coalescing by another type of interaction which would give a model reversible in time?

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- If $\xi = 0$, then particles coalesce.
- If $\xi(u) = \xi(v)$, then particles u and v coalesce after the meeting: because the drifts of $X(u,\cdot)$ and $X(v,\cdot)$ at time s are equal after the meeting

$$\xi(u) - \frac{1}{m(u,s)}\int_{\pi(u,s)}\xi(u)du = \xi(v) - \frac{1}{m(v,s)}\int_{\pi(v,s)}\xi(r)dr,$$

since $\pi(u,s) = \pi(v,s)$ for $X(u,s) = X(v,s)$.

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  since $\pi(u,s)=\pi(v,s)$ for $X(u,s)=X(v,s)$.
- If particle u is alone, i.e. $\pi(u,t)=\{u\}$, then it has no drift.

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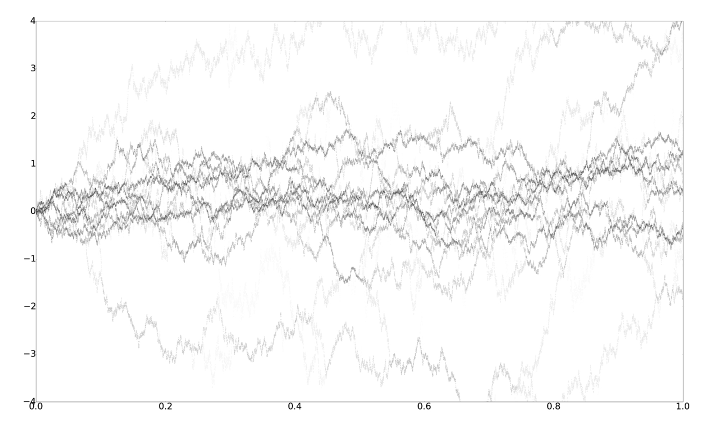
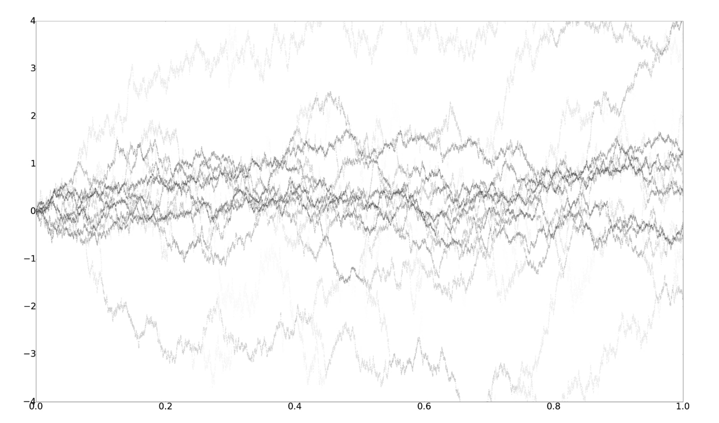


$$g(u) = 0, \quad \xi(u) = u, \quad u \in (0,1)$$



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The model is similar to the Howitt-Warren flow. The main difference is that in our case particles change the diffusion rate.
(Howitt, Warren '09; Schertzer, Sun, Swart '14)

SDE for the particle system

- One can show that $X(\cdot, t)$ is a continuous process in $L_2^{\uparrow} \subset L_2[0,1]$, if, e.g., $g, \xi \in L_{2+\varepsilon}$.

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$$X(u,\cdot) - \int_0^t \left(\xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(r)dr \right) ds \quad \text{is a continuous martingale}$$

and

$$\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$$

can be formally rewritten as

$$dX(u,t) = \frac{1}{m(u,t)} \int_0^1 \mathbb{I}_{\pi(u,t)} W(dr,dt) + \left(\xi(u) - \frac{1}{m(u,t)} \int_0^1 \mathbb{I}_{\pi(u,t)} \xi dr \right) dt,$$

where $\pi(u,t) = \{v : \; X(u,t) = X(v,t)\}$, $m(u,t) = \mathrm{Leb}(\pi(u,t))$, and W is a cylindrical Wiener process in $L_2[0,1]$.

SDE in $L_2^{\uparrow}$ for the particle system

The family of equations

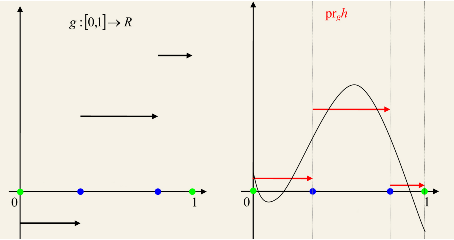
$$dX(u,t) = \frac{1}{m(u,t)} \int_0^1 \mathbb{I}_{\pi(u,t)} W(dr,dt) + \left(\xi(u) - \frac{1}{m(u,t)} \int_0^1 \mathbb{I}_{\pi(u,t)} \xi dr \right) dt,$$

can be rewritten as **one equation** but in $L_2^{\uparrow} \subset L_2[0,1]$:

$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t} \xi) dt \quad \text{in } L_2^{\uparrow},$$

where $X_t := X(\cdot, t)$ and pr_g is the **projection** in $L_2[0,1]$ onto

$$L_2(g) = \{ f : f \ \text{ is } \sigma(g)\text{-measurable} \}$$



Invariant measure and the Differential operator

- **Invariant measure on $L_2^{\uparrow}$:**

$$\Xi = \sum_{n=1}^{\infty} \Xi^n,$$

where Ξ^n is the distribution of $\sum_{k=1}^{n} \mathbb{I}_{[q_{k-1},q_k)} x_k$ and points of jumps $(q_1, \dots, q_{n-1})$ are distributed according to

$$d\nu_\xi^n = 2^{n-1} \prod_{k=1}^{n} (q_k - q_{k-1}) d\xi(q_1) \dots \xi(q_{n-1}), \quad \text{on} \quad 0 = q_0 < q_1 < \dots < q_n = 1$$

and values of jumps $(x_1, \dots, x_n)$ are uniformly distributed on $x_1 \le x_2 \le \dots \le x_n$.
One can show that $\operatorname{supp} \Xi = L_2^{\uparrow}(\xi) = \{f \in L_2^{\uparrow} : \ f \text{ is } \sigma(\xi)\text{-measurable}\}$

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- **Space of "smooth" functions:**

$$\mathcal{F}_0 = \left\{ U = u((h_1, \cdot), \ldots, (h_k, \cdot))\varphi(\| \cdot \|_{L_2}^2), \ u \in \mathcal{C}_b^2(\mathbb{R}^k), \ \varphi \in \mathcal{C}_0^2(\mathbb{R}), \ h_i \in L_2[0,1] \right\};$$

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- **Differential operator:** $\mathrm{D} U(g) = \operatorname{pr}_g \nabla^{L_2} U(g) \in L_2[0,1]$;

(Ex. $\mathrm{D} u((h,g)) = u'((h,g)) \operatorname{pr}_g h, \quad \mathrm{D}\|g\|_{L_2}^2 = 2g$)

Integration by parts and Dirichlet form

Integration by parts (K./ Renesse)

Let $U, V \in \mathcal{F}_0$. Then

$$\int_{L_2^\uparrow} (\mathrm{D}U(g), \mathrm{D}V(g))\Xi(dg) = -\int_{L_2^\uparrow} LU(g)V(g)\Xi(dg) - \int_{L^\uparrow 2} V(g)(\nabla^{L_2} U(g), \xi - \mathrm{pr}_g \xi)\Xi(dg).$$

(Examples $Lu((h,g)) = u''((h,g))\| \mathrm{pr}_g h\|_{L_2}^2, \quad L\|g\|_{L_2}^2 = 2\#g$)

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Dirichlet form:

$$\mathcal{E}(U,V) = \frac{1}{2}\int_{L_2^{\uparrow}(\xi)} (\mathrm{D}U(g), \mathrm{D}V(g))\Xi(dg), \quad U, V \in \mathcal{F}_0$$

Sticky-reflected particle system

Theorem (K./ Renesse)

$\mathcal{E}$ is a closable bilinear form on $L_2(L_2^{\uparrow}, \Xi)$, its closure is a quasi-regular local symmetric Dirichlet form and $\|\cdot\|_{L_2}$ is its intrinsic metric. Moreover, the associated Markov process X_t satisfies the following properties

- X_t solves
$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t} \xi)dt \quad \text{in } L_2^{\uparrow}[0,1]$$
- The process $\mu_t = X(\cdot, t)|_{\#} \mathrm{Leb}|_{[0,1]}$, that describes the evolution of particle mass, solves the equation
$$\frac{\partial}{\partial t}\mu_t = \frac{1}{2}\Delta\mu_t^* + \mathrm{div}(\sqrt{\mu_t}\dot{W}_t), \quad \text{in } \mathcal{P}_2(\mathbb{R}),$$
where $\mu_t^* = \sum_{x \in \mathrm{supp}\,\mu_t} \delta_x$
- $\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0,\nu)}{2t}}, \quad t \to +0.$

Thank you!

1. **Coalescing particle model**
   - *Konarovskyi, Renesse,* Modified Massive Arratia flow and Wasserstein diffusion, *Comm. Pure Appl. Math.* (2019)
   - *Konarovskyi,* A system of coalescing diffusion particles on $\mathbb{R}$, *Ann. Prob.* (2017)
   - *Konarovskyi,* On asymptotic behavior of the modified Arratia flow, *EJP* (2017)
   - *Konarovskyi, Marx,* On Conditioning Brownian Particles to Coalesce, *arXiv:2008.02568*
2. **Sticky-reflected particle model**
   - *Konarovskyi,* Coalescing-Fragmentating Wasserstein Dynamics: particle approach, *arXiv:1711.03011*
   - *Konarovskyi, Renesse,* Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line, *arXiv:1709.02839*
3. **Dean-Kawasaki equation**
   - *Lehmann, Konarovskyi, Renesse,* Dean-Kawasaki dynamics: Ill-posedness vs. Triviality, *Electron. Comm. Probab.* (2019)
   - *Konarovskyi, Lehmann, Renesse,* Dean-Kawasaki dynamics with smooth drift potential, *J. Stat. Phys.* (2020)