Sticky-reflected stochastic heat equation driven by colored noise

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Sticky-reflected stochastic heat equation on $[0, 1]$

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$$X_0 = g \geq 0, \quad X_t(0) = X_t(1) = 0$$

where $\lambda > 0$

It is similar to the SDE for sticky-reflected Brownian motion:

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}}dt + \mathbb{I}_{\{x(t)>0\}}dw(t),$$
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– only weak existence and uniqueness in law! (Engelbert and Peskir, 2014)

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Reason of investigation

- Sticky-reflected SHE vs. Reflected SHE (Nulart and Pardoux, 1992)

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + \delta_0(X_t) + \dot{W}_t$$

- Possible connection with wetting dynamics (Deuschel, Giacomin, Zambotti, 2004)
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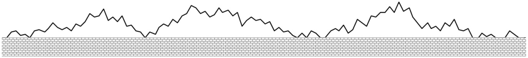

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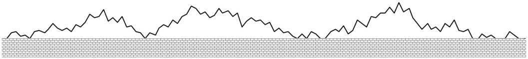

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Formulation of the main result

Sticky-reflected SHE:

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$$X_0 = g \geq 0, \quad X_t(0) = X_t(1) = 0,$$

where Q is non-negative definite self-adjoint Hilbert-Schmidt operator in $L_2[0,1]$

Solution to sticky-reflected SHE

A continuous process $X : [0,\infty) \times [0,1] \to \mathbb{R}$ is called a **weak solution** to the sticky-reflected SHE if for any $\varphi \in C^2[0,1]$ with $\varphi(0) = \varphi(1) = 0$

$$M_t^{\varphi} := \langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \frac{1}{2}\int_0^t \langle X_s, \varphi''\rangle\, ds - \int_0^t \left\langle \lambda \mathbb{I}_{\{X_s=0\}}, \varphi\right\rangle ds$$

is a martingale with quadratic variation

$$[M^{\varphi}]_t = \int_0^t \|Q(\mathbb{I}_{\{X_s>0\}}\varphi)\|_{L_2}^2\, ds.$$

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Let $\{e_k,\ k \geq 1\}$ and $\{\mu_k,\ k \geq 1\}$ be eigenvectors and eigenvalues of Q. Define

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Theorem K. 2021

If $\chi^2 > 0$ a.e., then the sticky-reflected SHE admits a weak solution.

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Meaning of assumtion $\chi^2 > 0$

The equation

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} dw(t)$$
$$x(0) = 0$$

has no solution

$\chi^2 = \sum_{k=1}^{\infty} \mu_k^2 e_k^2 > 0$ means that the solution X to the equation

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Description of the idea of construction of solution

using the equation

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} dw(t)$$

Approximating sequence

Consider the SDE for sticky-reflected BM:

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We approximate its solution by the solutions to the SDEs

$$dx_n(t) = \lambda(1 - \kappa_n^2(x_n(t)))dt + \kappa_n(x_n(t))dw(t),$$
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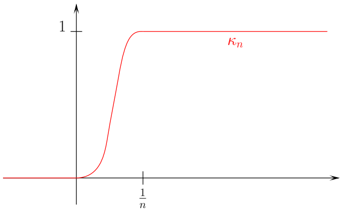
which have non-negative strong solutons $x_n(t) \geq 0$.



$$\kappa_n(y) \to \mathbb{I}_{\{y>0\}},$$
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for $y \geq 0$, as $n \to \infty$.

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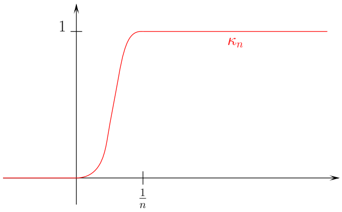

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Problem of approximation

One can show that $\{x_n,\ n \geq 1\}$ is tight in $C[0,T]$ and $\{\kappa_n^2(x_n),\ n \geq 1\}$ is tight in $L^2[0,T]$ in weak topology

$$\Downarrow$$

$$x_n \to x \quad \text{in} \quad C[0,T] \quad \text{and} \quad \kappa_n^2(x_n) \to \rho^2 \quad \text{in} \quad L_2[0,T] \quad \text{in weak t.}$$

along a subsequence.

But

$$M_n(t) := x_n(t) - x_0 + \int_0^t \lambda(1 - \kappa_n^2(x_n(s)))ds$$

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Why $\rho^2(s) = \mathbb{I}_{\{x(s)>0\}}$?

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Why $\rho^2(s) = \mathbb{I}_{\{x(s)>0\}}$?

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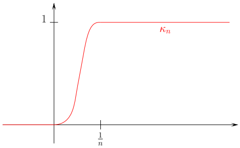
- $\kappa_n^2(y_n) \not\to \mathbb{I}_{\{y>0\}}$ as $y_n \to y$.



- If x is a continuous non-negative semimartingale with q.v.

$$[x]_t = \int_0^t \sigma^2(s)ds,$$

then $[x]_t = \int_0^t \mathbb{I}_{\{x(s)>0\}}\sigma^2(s)ds$.

Proof.

$$\int_0^t \sigma^2(s)\mathbb{I}_{\{x(s)=0\}}ds = \int_0^t \mathbb{I}_{\{0\}}(x(s))d[x]_s = \int_{-\infty}^{+\infty} \mathbb{I}_{\{0\}}(y)L_t^y dx = 0,$$

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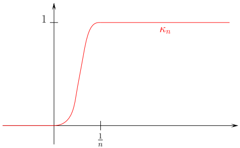

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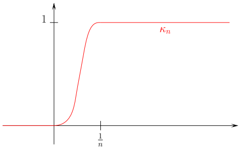

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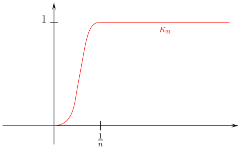

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Identification of quadratic variation

Remind

$$M_n(t) := x_n(t) - x_0 + \int_0^t \lambda(1 - \kappa_n^2(x_n(s)))ds$$

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$$M(t) := x(t) \quad - x_0 \quad + \int_0^t \lambda(1 - \rho^2(s))ds$$

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$$[x]_t = [M]_t = \int_0^t \rho^2(s)ds = \int_0^t \mathbb{I}_{\{x(s)>0\}}\rho^2(s)ds$$

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Proof of existence of solution to

sticky-reflected SHE

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + \lambda \mathbb{I}_{\{X_t=0\}} + \mathbb{I}_{\{X_t>0\}} Q \dot{W}_t$$

Discrete equation

We discretize only the space variable $u \in [0,1]$ by $\frac{k}{n}$, $k = 1, \ldots, n$.
Set $\pi_k^n = \mathbb{I}_{\left[\frac{k-1}{n}, \frac{k}{n}\right)}$ and define

$$w_k^n(t) := \sqrt{n} \int_0^t \int_0^1 (Q\pi_k^n)(u) W(du, ds)$$

Consider the following SDE

$$dx_k(t) = \frac{1}{2}\Delta^n x_k(t)dt + \mathbb{I}_{\{x_k(t)=0\}}dt + \sqrt{n}\mathbb{I}_{\{x_k(t)>0\}}dw_k^n(t), \quad k = 1, \ldots, n,$$

with $x_0(t) = x_{n+1}(t) = 0$ and $\Delta^n x_k = n^2 \left(x_{k+1} + x_{k-1} - 2x_k\right)$

Tightness of linear approximation

Define

$$\tilde{X}_t^n(u) = (un - k + 1)x_k^n(t) + (k - nu)x_{k-1}^n(t), \quad t \in [0, T], \quad u \in \pi_k^n.$$

Then $\tilde{X}^n \geq 0$ and $X^n \in C([0,T], C[0,1])$ a.s.

Theorem. Funaki '83

The family of processes $\{\tilde{X}^n, \ n \geq 1\}$ is tight in $C([0,T], C[0,1])$

Idea of Proof.

$$\tilde{X}_t^n(u) = \int_0^1 p^n(t,u,v)g(v)dv + \lambda \int_0^t \int_0^1 p^n(t-s,u,v)\mathbb{I}_{\{\tilde{X}_s^n(\lceil v \rceil)=0\}}dsdv$$
$$+ \int_0^t \int_0^1 p^n(t-s,u,v)\mathbb{I}_{\{\tilde{X}_s^n(\lceil v \rceil)>0\}}QdW_s du$$

By properties of the discrete heat semigroup p^n, for every $\gamma > 0$, $T > 0$,

$$\mathbb{E}\left[(\tilde{X}_t^n(u))^\gamma\right] \leq C \quad \text{and} \quad \mathbb{E}\left[|\tilde{X}_t^n(u) - \tilde{X}_s(v)|^{2\gamma}\right] \leq C\left(|t-s|^{\frac{\gamma}{2}} + |u-v|^{\frac{\gamma}{2}}\right)$$

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$$\tilde{X}_t^n(u) = \int_0^1 p^n(t, u, v) g(v) dv + \lambda \int_0^t \int_0^1 p^n(t - s, u, v) \mathbb{I}_{\{\tilde{X}_s^n(\lceil v \rceil) = 0\}} ds dv + \int_0^t \int_0^1 p^n(t - s, u, v) \mathbb{I}_{\{\tilde{X}_s^n(\lceil v \rceil) > 0\}} Q dW_s du$$

By properties of the discrete heat semigroup p^n, for every $\gamma > 0$, $T > 0$,

$$\mathbb{E}\left[(\tilde{X}_t^n(u))^\gamma\right] \leq C \quad \text{and} \quad \mathbb{E}\left[|\tilde{X}_t^n(u) - \tilde{X}_s(v)|^{2\gamma}\right] \leq C\left(|t - s|^{\frac{\gamma}{2}} + |u - v|^{\frac{\gamma}{2}}\right)$$

Tightness of linear approximation

Define

$$\tilde{X}_t^n(u) = (un - k + 1)x_k^n(t) + (k - nu)x_{k-1}^n(t), \quad t \in [0, T], \quad u \in \pi_k^n.$$

Then $\tilde{X}^n \geq 0$ and $X^n \in C([0, T], C[0, 1])$ a.s.

Theorem. **Funaki '83**

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Martingale problem for X^n

Set

$$X_t^n(u) = x_k(t), \quad \frac{k-1}{n} \le u < \frac{k}{n}, \quad u \in [0,1].$$

Remark that $X^n \ge 0$.

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$$M_t^{n,\varphi} := \langle X_t^n, \varphi \rangle - \langle X_0^n, \varphi \rangle - \frac{1}{2}\int_0^t \left\langle X_s^n, \tilde{\Delta}^n \varphi \right\rangle ds - \lambda \int_0^t \left\langle \mathbb{I}_{\{X_s^n = 0\}}, \varphi \right\rangle ds$$

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$$[M^{n,\varphi}]_t = \int_0^t \left\| Q(\mathbb{I}_{\{X_s^n > 0\}} \varphi) \right\|^2 ds \to \int_0^t \left\| Q(\sigma_s \varphi) \right\|^2 ds = [M^\varphi]_t$$

Equivalently

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + \lambda(1 - \sigma_t) + \sigma_t Q \dot{W},$$

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Identification of coefficient σ

Proposition (K., 2021)

Let X solves the equation

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\frac{\partial^2 X_t}{\partial u^2} + a_t + \sigma_t Q\dot{W},$$

$$X_t(0) = X_t(1) = 0, \quad X_0(u) = g(u)$$

and $X \geq 0$. Then $\sigma_t = \mathbb{I}_{\{X_t > 0\}}\sigma_t$.

Identification of the coefficients

We come back to our equation with undefined coefficients:

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By the previous proposition,

$$\sigma_t(u) = \mathbb{I}_{\{X_t(u)>0\}}\sigma_t(u) = \lim_n \mathbb{I}_{\{X_t(u)>0\}}\mathbb{I}_{\{X_t^n(u)>0\}} = \mathbb{I}_{\{X_t(u)>0\}}$$

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Idea of proof of the key proposition

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Ito's formula

Lemma. (Ito's formula)

Assume that $F \in C^2(\mathbb{R})$ has a bounded second derivative. Then

$$\langle F(X_t), 1\rangle = \langle F(X_0), 1\rangle - \frac{1}{2}\int_0^t \left\langle F''(X_s)\dot{X}_s, \dot{X}_s \right\rangle ds + \int_0^t \langle F'(X_s), a_s\rangle\, ds$$
$$+ \frac{1}{2}\int_0^t \langle Q[\sigma_s F''(X_s)\cdot], Q[\sigma_s \cdot]\rangle_{HS}\, ds + M_F(t),$$

where

$$[M_F]_t = \int_0^t \|Q[\sigma_s F'(X_s)]\|^2 ds$$

and

$$\dot{X}_t = \sum_{k=1}^{\infty} \langle X_t, e_k\rangle e_k', \quad e_k(u) = \sqrt{2}\sin \pi k u$$

Idea of Proof. Idea of proof: Apply usual Ito's formula to $\langle F(Z_t^n), 1\rangle$ for $Z_t^n = \sum_{k=1}^n \langle X_t, e_k\rangle e_k$

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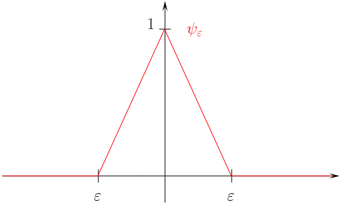
Proof of the key proposition

Take

$$F_\varepsilon(x) := \int_{-\infty}^{x} \int_{-\infty}^{y} \psi_\varepsilon(r)\,dy\,dr,$$



$$0 \le F'_\varepsilon(x) \le 2\varepsilon, \quad F''_\varepsilon(x) \to \mathbb{I}_{\{0\}}(x)$$

Apply Ito's formula to F_ε:

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Hence all green terms $\to 0$ and red term $\to \int_0^t \left\langle Q[\sigma_s \mathbb{I}_{\{X_s=0\}}\cdot], Q[\sigma_s\cdot]\right\rangle_{HS} ds$

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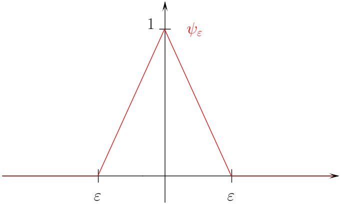

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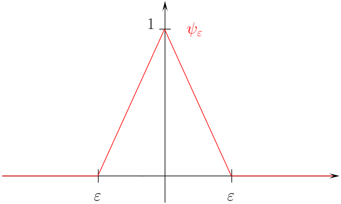

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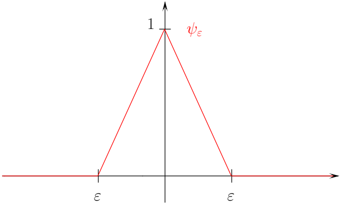

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Open problem and references

Open problems:

- Is a solution to the equation unique?
- Does the solution of the equation with the identity operator Q exists?
- What is the invariant measure for the dynamics?
- How much time does the equation spend at zero?

Vitalii Konarovskyi,
Sticky-Reflected Stochastic Heat Equation Driven by Colored Noise
Ukrain. Math. J., Vol. 72, no. 9, 2021
(arXiv:2005.11773)

Vitalii Konarovskyi,
Coalescing-Fragmentating Wasserstein Dynamics: particle approach
(arXiv:1711.03011)

Thank you for your attention!