A particle model for Wasserstein type diffusion

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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Dean-Kawasaki Equation

Systems of interacting particles in random environment

Consider a system of SDEs in $\mathbb{R}^d$

$$dx_i(t) = -\sum_{j=1}^{n} \nabla V(x_i(t) - x_j(t))dt + \sqrt{n} dw_i(t)$$
$$x_i(0) = x_i^0, \qquad i = 1, \dots, n,$$

where w_i are independent Brownian motions and V is an interaction potential

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Evolution of particle mass

Let $V = 0$ (no interaction) and $d = 1$. Take

$$\mu_t := \sum_{i=1}^{n} \frac{1}{n} \delta_{x_i(t)} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\sqrt{n} w_k(t)}, \quad t \geq 0.$$

Use Ito's formula to $\langle \varphi, \mu_t \rangle = \int_R \varphi d\mu_t = \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i(t))$:

$$\begin{aligned} d \langle \varphi, \mu_t \rangle &= \frac{1}{2n} \sum_{i=1}^{n} \varphi''(x_i(t)) n dt + \frac{1}{n} \sum_{i=1}^{n} \varphi'(x_i(t)) dx_i(t) \\ &= \frac{n}{2} \langle \varphi'', \mu_t \rangle dt + dM_t^{\varphi} \end{aligned}$$

where M_t^{φ} is a martingale with q.v.

$$[M^{\varphi}]_t = \frac{n}{n^2} \int_0^t \sum_{i=1}^{n} \varphi'(x_i(s))^2 dt = \int_0^t \langle (\varphi')^2, \mu_s \rangle ds$$

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Dean-Kawasaki equation

For every $\varphi \in C_b^2(\mathbb{R})$ the process $\mu_t = \frac{1}{n}\sum_{i=1}^n \delta_{\sqrt{n}w_i(t)}$ satisfies

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Formally, μ_t solves the equation

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta \mu_t + \nabla \cdot \left(\sqrt{\mu_t}\dot{W}_t \right)$$

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- **Dean-Kawasaki equation** for $\alpha = n$ and $F(\mu) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} V(x-y)\mu(dx)\mu(dy)$

$\frac{\delta F(\mu)}{\delta \mu}(x) = \lim_{\varepsilon \to 0+} \frac{F(\mu + \varepsilon \delta_x) - F(\mu)}{\varepsilon} = \int_{\mathbb{R}} V(x-y)\mu(dy)$, if $V(x) = V(-x)$

The Dean-Kawasaki equation

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The equation is used for modeling of behaviour of huge number of particles in the Langevin dynamics.

(K. Kawasaki '94; D. Dean '96; A. Donev, E. Vanden-Eijnden '14, '15; B. Derrida '16; J. Zimmer '19; B. Gess '19)

F corresponds for the interaction between particles

Does the D-K equation have solutions for every $\alpha > 0$, any initial condition μ_0 and interaction potential F.

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where

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Definition of solution to the Dean-Kawasaki equation

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Definition of (martingale) solution

A continuous process μ_t, $t \geq 0$ is a solution to (DK_F^α eq) if, for every $\varphi \in \mathrm{C}_b^2(\mathbb{R}^d)$

$$\langle\varphi,\mu_t\rangle = \langle\varphi,\mu_0\rangle + \frac{\alpha}{2}\int_0^t \langle\Delta\varphi,\mu_s\rangle ds - \int_0^t \left\langle \nabla\varphi\cdot\nabla\frac{\delta F(\mu_s)}{\delta\mu_s},\mu_s\right\rangle ds + M_t^\varphi$$

where M^φ is a martingale with quadratic variation

$$\int_0^t \langle|\nabla\varphi|^2,\mu_s\rangle ds.$$

Well-posedness of Dean-Kawasaki equation

Theorem (K., T. Lehmann, M. von Renesse)

Let $\mu_0(\mathbb{R}^d) = 1$, and F be smooth and bounded. Then the equation

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla \cdot \left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla \cdot \left(\sqrt{\mu_t}\dot{W}_t\right)$$

has a (unique) solution iff $\alpha = n$ and $\mu_0 = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i^0}$. Moreover, it is defined as above:

$$\mu_t = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i(t)}.$$

Elect. Comm. Probab '19 for $F = 0$; J. Stat. Phys. '20 for F smooth

Singular interaction potential

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \Gamma(\mu_t) + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

To have the equation which has no trivial solutions,
a singular Γ is needed!

There is known a singular Γ such that the D-K equation has a solution μ_t which is the **Wasserstein diffusion** that is a Markov process with some invariant measure. **(von Renesse, Sturm '09)**

Aim of my talk: We are going to use a particle approach in order to have another models which can solve the D-K equation (with another Γ).

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Modified Massive Arratia Flow (on $\mathbb{R}$)

Some observation

Let $w_1,\ w_2$ be independent Brownian motions on $\mathbb{R}$ with diffusion rates $a_1,\ a_2$

$$\mu_t := m_1\delta_{w_1(t)} + m_2\delta_{w_2(t)}$$

By the Ito formula:

$$\begin{aligned}\langle \varphi, \mu_t\rangle =& m_1\varphi(w_1(t)) + m_2\varphi(w_2(t)) \\ =& \frac{1}{2}\text{bdd. variation} + \int_0^t [m_1\dot\varphi(w_1(s))dw_1(s) + m_2\dot\varphi(w_2(s))dw_2(s)]\end{aligned}$$

So, its quadratic variation is

$$\int_0^t \left(m_1^2\dot\varphi(w_1(s))^2 a_1 + m_2^2\dot\varphi(w_2(s))^2 a_2\right) ds = \int_0^t \langle \dot\varphi^2, \mu_s\rangle ds,$$

if $a_1 = \frac{1}{m_1}$ and $a_2 = \frac{1}{m_2}$.

The diffusion rate of each particle has to be inversely proportional to its mass!

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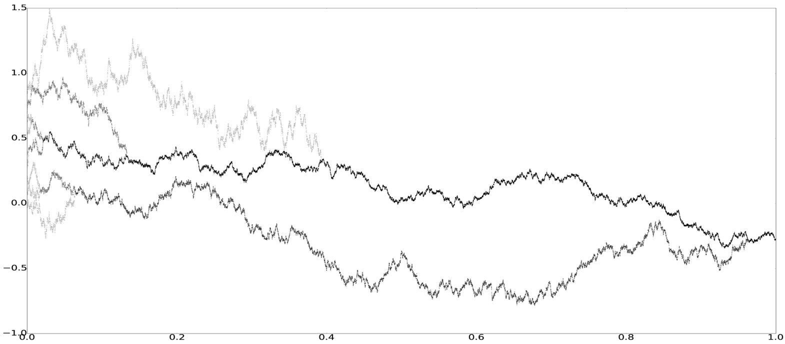
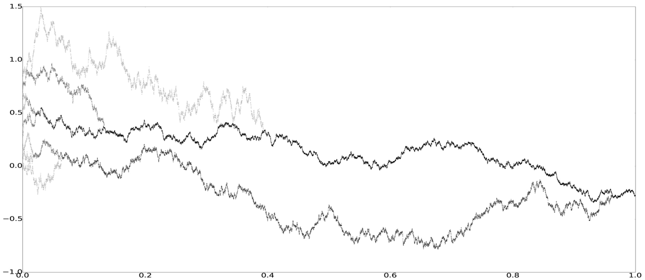
n-particle system

Consider n particle system on $\mathbb{R}$ such that

- particles start from points $\frac{i}{n}$, $i = 1, \ldots, n$ with masses $\frac{1}{n}$ and move as Brownian motions;
- diffusion rate of each particle inversely depends on its mass;
- particles move independently of each other and coalesce after meeting.



n-particle system as a family of martingales



Let $x_i(t)$ be the position of particle at time t starting from $\frac{i}{n}$, $i = 1, \ldots, n$ then

1. x_i is a continuous square integrable martingale for all i.
2. $x_i(0) = \frac{i}{n}$;
3. $x_i(t) \leq x_j(t)$, $i < j$, $t \geq 0$;
4. $[x_i]_t = \int_0^t \frac{ds}{m_i(s)}$,
   where $m_i(t) = \frac{1}{n}|\{j : \ x_i(t) = x_j(t)\}|$;
5. $[x_i, x_j]_t = 0$, $t < \tau_{i,j}$,
   where $\tau_{i,j} = \inf\{t : \ x_i(t) = x_j(t)\}$.

Infinite particle system

Set

$$X_n(u,t) = \sum_{i=1}^{n} x_i^n(t)\mathbb{I}_{\left[\frac{i-1}{n},\frac{i}{n}\right]}(u), \quad u \in [0,1], \quad t \geq 0;$$

Theorem (K., Ann. Probab. '17)

There exists a subsequence X_{n_k}, $k \geq 1$, which converges to a process X which satisfies the following properties

1. $X(u,\cdot)$ is continuous matringale for all u;
2. $X(u,0) = u$, $u \in [0,1]$;
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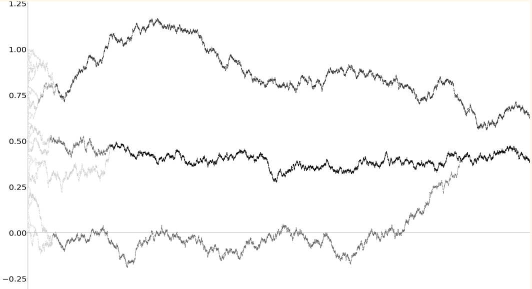

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   where $\tau_{u,v} = \inf\{t : \ X(u,t) = X(v,t)\}$.

Open problems:

1. Does the sequence X_n, $n \geq 1$, converges to X?
2. Does Conditions 1.-5. uniquely determine the distribution of X?

Some basic properties of modified massive Arratia flow

Let $T > 0$.

1. Let $N(t)$ be a number of distinct particles at time t. Then

$$\mathbb{E}\frac{1}{m(u,t)} \le \frac{C}{\sqrt[3]{t}}, \quad u \in [0,1], \quad t \in [0,T]$$

and hence

$$\mathbb{E}N(t) = \mathbb{E}\int_0^1 \frac{du}{m(u,t)} \le \frac{C}{\sqrt[3]{t}}, \quad t \in [0,T];$$

2. The process $X(\cdot,t)$, $t \ge 0$, takes values in

$$L_2^{\uparrow} = \{g \in L_2[0,1] : \ g \text{ is non-decreasing}\}$$

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LDP for modified massive Arratia flow

Let $w(t)$, $t \in [0,T]$, be a Brownian motion in $\mathbb{R}^d$ starting at x_0 and $w_\varepsilon(t) = w(\varepsilon t)$. Then $\{w_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP in $\mathrm{C}([0,T],\mathbb{R}^d)$ with the rate function

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot\varphi(t)\|^2_{\mathbb{R}^d} dt, & \varphi \in H^2_{x^0}([0,T],\mathbb{R}^d), \\ +\infty, & \text{otherwise.} \end{cases}$$

Roughly speaking

$$\mathbb{P}\{w_\varepsilon \in B_r(\psi)\} \sim e^{-\frac{1}{\varepsilon}\inf\limits_{B_r(\psi)} I}, \quad \varepsilon \to 0+.$$

Theorem (K., M. von Renesse, Comm. Pure Appl. Math. '19)

The family $X_\varepsilon = \{X(u,\varepsilon t),\ u \in [0,1], t \in [0,T]\}$, $\varepsilon > 0$, satisfies the LDP in $\mathrm{C}([0,T],L_2^\uparrow)$ with rate function

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A consequence from LDP

Let $w(t)$, $t \in [0,T]$, be a standard Brownian motion in $\mathbb{R}^d$ starting at x_0.
Let $T = 1$. Then $w_\varepsilon(1) = w(\varepsilon)$ satisfies the LDP in $\mathbb{R}^d$:

$$\mathbb{P}\{w(\varepsilon) \in B_r(y)\} \sim e^{-\frac{1}{\varepsilon} \inf_{x \in B_r(y)} \frac{\|x_0 - x\|^2_{\mathbb{R}^d}}{2}}, \quad \varepsilon \to 0+ .$$

Varadhan formula (Varadhan, CPAM '87):
If $w(t)$, $t \geq 0$, is a Brownian motion of a Riemannian manifold, then

$$\mathbb{P}\{w(\varepsilon) \in B_r(y)\} \sim e^{-\frac{1}{\varepsilon} \inf_{x \in B_r(y)} \frac{d(x_0,x)^2}{2}}, \quad \varepsilon \to 0+ .$$

where d is the geodesic distance.

Corollary

The family $X(\cdot,\varepsilon)$, $\varepsilon > 0$, satisfies the LDP in $L_2^\uparrow$:

$$\mathbb{P}\{X(\cdot,\varepsilon) \in B_r(f)\} \sim e^{-\frac{1}{\varepsilon} \inf_{x \in B_r(f)} \frac{\|\mathrm{id} - g\|^2_{L_2}}{2}}, \quad \varepsilon \to 0+ .$$

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We consider the evolution of particle mass in the modified massive Arratia flow:

$$\mu_t = X(\cdot, t)_{\#} \operatorname{Leb}_1, \quad t \geq 0,$$

where $\operatorname{Leb}_1 = \operatorname{Leb}|_{[0,1]}$.

Theorem (K., M. von Renesse, Comm. Pure Appl. Math. '19)

1. The process μ_t, $t \geq 0$, solves the equation

$$d\mu_t = \frac{1}{2}\Delta\mu_t^* dt + \operatorname{div}(\sqrt{\mu_t} dW_t),$$

where $\mu_t^* = \sum_{x \in \operatorname{supp} \mu_t} \delta_x$.

2. The Varadhan formula:

$$\mathbb{P}\{\mu_\varepsilon \in B_r(\nu)\} \sim e^{-\frac{1}{\varepsilon} \inf_{\rho \in B_r(\nu)} \frac{d_{\mathcal{W}}^2(\operatorname{Leb}_1, \rho)}{2}}, \quad \varepsilon \to 0+$$

where $d_{\mathcal{W}}$ denotes the Wasserstein distance on the space of probability measures $\mathcal{P}_2(\mathbb{R})$ on $\mathbb{R}$ with finite second moment.

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Thank you!