Stochastic Block Model in a new critical regime and the Interacting Multiplicative Coalescent

Vitalii Konarovskyi[*,†]

[*]Leipzig University

[†]Institute of Mathematics of NAS of Ukraine

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joint work with Vlada Limic

UNIVERSITÄT LEIPZIG

NATIONAL ACADEMY OF SCIENCES OF UKRAINE
INSTITUTE OF MATHEMATICS

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Stochastic Block Model $G(n, p, q)$ is a random graph such that:

- consists of nm vertices divided into m subsets $(m = 2)$;
- edges are drown independently;
- intra class edges appear with probability $p = p_n$;
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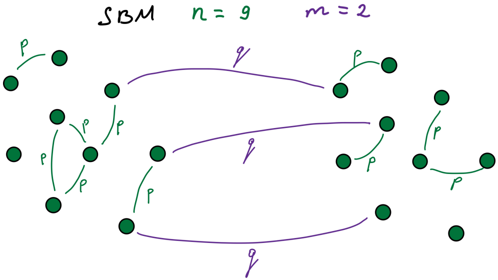

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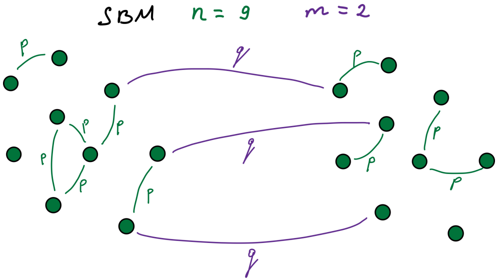

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Largest Component of SBM

$C_1(n)$ – the largest component of the SBM

It is well-known:

- If $p_n = q_n = \frac{a}{mn}$, then SBM is an Erdős-Rényi graph for which:
  - for $a > 1$, $C_1(n) \sim \Theta(n)$;
  - for $a < 1$, $C_1(n) \sim \Theta(\ln n)$; (Erdős, Rényi '60, '61)
  - for $a = 1$, $C_1(n) \sim \Theta(n^{2/3})$.
- If $p_n = \frac{a}{mn}$, $q_n = \frac{b}{mn}$, then
  - $a + (m-1)b > m$, $C_1(n) \sim \Theta(n)$;
  - $a + (m-1)b \leq m$, $C_1(n) \sim o(n)$. (Bollobás, Janson, Riordan '07)

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Scaling limit of Erdős-Rényi Graphs

$G(n,p)$ – a Erdős-Rényi random graph with n vertices and edges appearing with prob. $p = p_n(t) = \frac{1}{n} + \frac{t}{n^{4/3}}$.

$\mathrm{ER}_n(t) := \frac{1}{n^{2/3}}(C_1(n,t), C_2(n,t), \ldots, C_k(n,t), 0, 0, \ldots\ldots)$, where $C_k(n,t)$ is the k-th largest component.

Theorem. (Aldous '97, Anmerdariz '01, Limic '98,'19)

For every $t \in \mathbb{R}$ the sequence $\mathrm{ER}_n(t)$ converges in $l^2_\downarrow$ to $X^*(t)$ in distribution, where $X^*(t)$ is the ordered sequence of excursions of

$$W(s) - \frac{1}{2}s^2 + ts, \quad s \geq 0,$$

above past minima. $X^*(t)$, $t \in \mathbb{R}$, is called the *standard Multiplicative coalescent*, and is a Markov process in $l^2_\downarrow$.

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Interacting Multiplicative Coalescent and Main Result

$$\mathrm{SBM}_n(t,s) := \frac{1}{n^{2/3}}(C_1(n,t), C_2(n,t), \dots, C_k(n,t), 0, 0), \quad t \in \mathbb{R}, \quad s \geq 0,$$

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$$q = q_n(t) = \frac{s}{n^{4/3}}, \quad p = p_n(t) = \frac{1}{n} + \frac{t}{n^{4/3}}.$$

For $s \geq 0$ and a fixed family of indep. r.v. $\xi_{i,j} \sim \mathrm{Exp}(1)$, $i, j \geq 1$, define a random map $\mathrm{RMM}_s : l^2_\downarrow \times l^2_\downarrow \to l^2_\downarrow$:

- consider coord. of $x, y \in l^2_\downarrow$ as a masses of corresponding vertices of a graph;
- for every $i, j \geq 1$ draw an edge between x_i and y_j iff $\xi_{i,j} \leq s x_i y_j$;
- define $\mathrm{RMM}_s(x,y)$ as a vector of the ordered masses of connected components.

Theorem. (K., Limic '20)

For every $t \in \mathbb{R}$ and $s \geq 0$ the process $\mathrm{SBM}_n(t,s)$ converges in $l^2_\downarrow$ in distribution to $\mathrm{RMM}_s(X^*(t), Y^*(t))$, where X^*, Y^* are independent standard multiplicative coalescents that are independent of ξ

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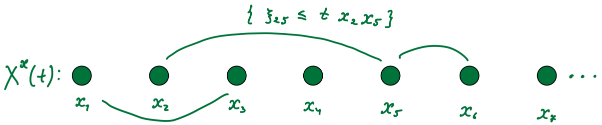
Idea of Proof. ER_n and Multiplicative Coalescent

For $x \in l^2_{\downarrow}$, and independent $\xi_{i,j} \sim \mathrm{Exp}(1)$ define



$X^x(t)$ – ordered masses of connected componnents, $t \geq 0$.

The process $X^x(t)$, $t \geq 0$, is a Markov process in $l^2_{\downarrow}$ started from x. Moreover, it evolves according to the multiplicative coalescent dynamics:

each pair of blocks of mass a and b merges at rate ab

into a single block of mass $a + b$

Since for $x = \left(n^{-2/3}, \dots, n^{-2/3}, 0, 0, \dots\right)$ and some $t_n \to t$, $n \to \infty$,

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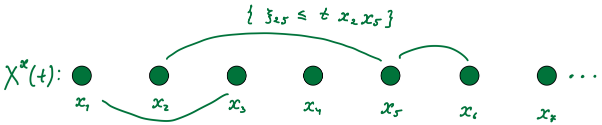

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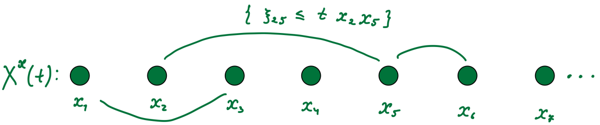

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$x, y \in l^2_{\downarrow}$ and $\xi'_{i,j}, \xi''_{i,j}, \xi'''_{i,j} \sim \mathrm{Exp}(1)$, $i, j \geq 1$, are independent

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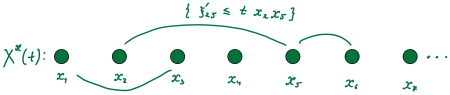

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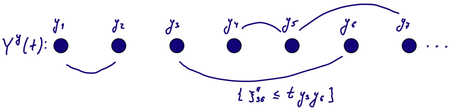

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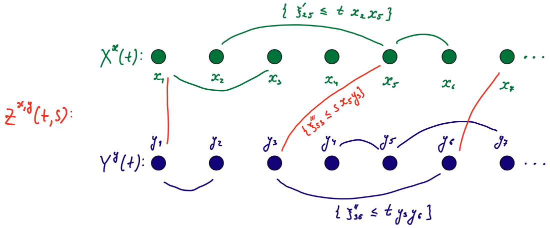

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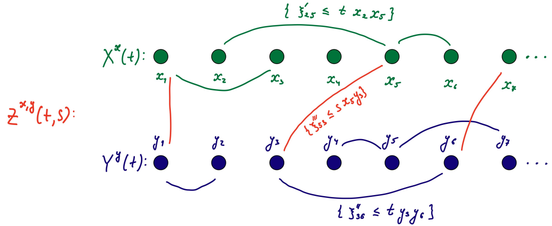

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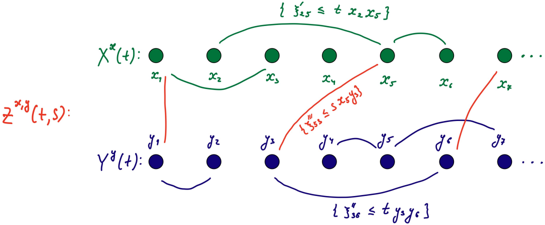

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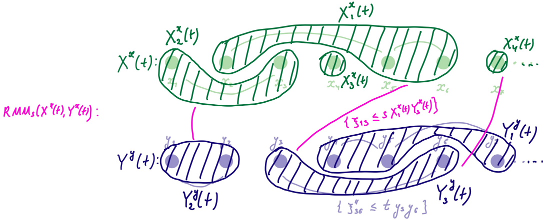

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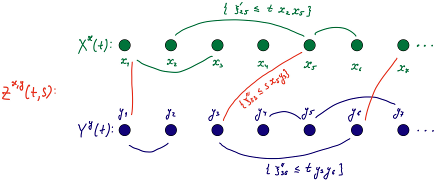

Let $x = y = \left(n^{-2/3}, \ldots, n^{-2/3}, 0, 0, \ldots\right)$. For some $t_n \to t$, $n \to \infty$,

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and for some $s_n \to s$, $n \to \infty$,

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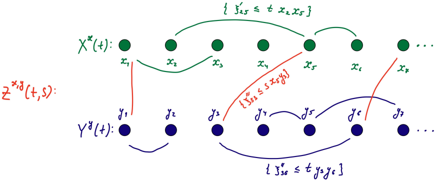

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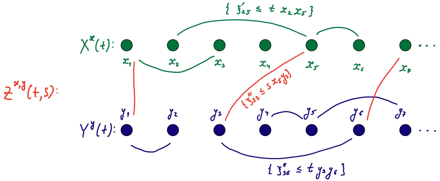

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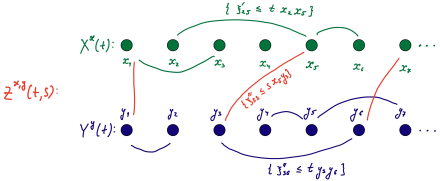

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$$p_n = \frac{1}{n} + \frac{t}{n^{4/3}} = 1 - e^{(t_n + n^{1/3})n^{-2/3}n^{-2/3}} = \mathbb{P}\left\{\xi_{i,j} \le (t_n + n^{1/3})x_i x_j\right\}$$

and for some $s_n \to s$, $n \to \infty$,

$$q_n = \frac{s}{n^{4/3}} = 1 - e^{s_n n^{-2/3} n^{-2/3}} = \mathbb{P}\left\{\xi_{i,j} \le s_n x_i y_j\right\}$$

Hence,

$$\mathrm{SBM}_n(t,s) \stackrel{d}{=} Z^{x,y}(t_n + n^{1/3}, s_n) \stackrel{d}{=} \mathrm{RMM}_{s_n}(X^x(t_n + n^{1/3}), Y^y(t_n + n^{1/3})).$$

Idea of proof. Continuity of RMM

The convergence

$$\mathrm{SBM}_n(t,s) \overset{d}{=} \mathrm{RMM}_{s_n}(X^x(t_n+n^{1/3}), Y^y(t_n+n^{1/3})) \to \mathrm{RMM}_s(X^*(t), Y^*(t)),$$

follows from

$$\left(\mathrm{ER}_n(t) \overset{d}{=}\right) \quad X^x(t_n+n^{1/3}) \xrightarrow{d} X^*(t), \quad Y^x(t_n+n^{1/3}) \xrightarrow{d} Y^*(t), \quad n\to\infty$$

and

Proposition (K., Limic '20)

Let $x^n \to x$, $y^n \to y$ in $l^2_\downarrow$ and $s_n \to s$ in $[0,\infty)$. Then

$$\mathrm{RMM}_{s_n}(x_n, y_n) \to \mathrm{RMM}_s(x,y) \quad \text{in } l^2_\downarrow \text{ in probability}$$

as $n\to\infty$.

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References

V. Konarovskyi, V. Limic
Stochastic Block Model in a new critical regime and the Interacting Multiplicative Coalescent.
arXiv:2003.10958

Thank you for your attention!