Conditional Distribution of Independent Brownian Motions to Event of Coalescing Paths

Vitalii Konarovskyi[*][†]

[*]Leipzig University

[†]Institute of Mathematics of NAS of Ukraine

Malliavin Calculus and its Applications, 2020

joint work with Victor Marx

UNIVERSITÄT LEIPZIG

NATIONAL ACADEMY OF SCIENCES OF UKRAINE
INSTITUTE OF MATHEMATICS

Simple observation

Observation: Let W_1, W_2 be independent standard Brownian motions on $\mathbb{R}$.

$$\mathbb{P}\{W_1 \in A | W_1 = W_2\} = \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A \middle| \frac{W_1 - W_2}{2} = 0\right\}$$
$$= \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A\right\}$$

The conditional distribution of the standard Brownian motion W_1 to the event $\{W_1 = W_2\}$ is the distribution of Brownian motion with diffusion rate $\frac{1}{2}$

Goal: Find the conditional distribution

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\},$$

where $\mathbf{Coal}$ is the set of coalescing paths and

1. $X(t) = (W_1(t), \ldots, W_n(t))$, $t \geq 0$, and W_k are independent Brownian motions;
2. $X(u,t) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, and W is a cylindrical Wiener process in $L_2 := L_2[0,1]$.

Simple observation

Observation: Let W_1, W_2 be independent standard Brownian motions on $\mathbb{R}$.

$$\mathbb{P}\{W_1 \in A | W_1 = W_2\} = \mathbb{P}\left\{ \frac{W_1 + W_2}{2} \in A \middle| \frac{W_1 - W_2}{2} = 0 \right\}$$
$$= \mathbb{P}\left\{ \frac{W_1 + W_2}{2} \in A \right\}$$

The conditional distribution of the standard Brownian motion W_1 to the event $\{W_1 = W_2\}$ is the distribution of Brownian motion with diffusion rate $\frac{1}{2}$

Goal: Find the conditional distribution

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\},$$

where $\mathbf{Coal}$ is the set of coalescing paths and

1. $X(t) = (W_1(t), \dots, W_n(t))$, $t \geq 0$, and W_k are independent Brownian motions;
2. $X(u, t) = u + W_t(u)$, $u \in [0, 1]$, $t \geq 0$, and W is a cylindrical Wiener process in $L_2 := L_2[0, 1]$.

Simple observation

Observation: Let W_1, W_2 be independent standard Brownian motions on $\mathbb{R}$.

$$\mathbb{P}\left\{W_1 \in A | W_1 = W_2\right\} = \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A \middle| \frac{W_1 - W_2}{2} = 0\right\}$$
$$= \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A\right\}$$

The conditional distribution of the standard Brownian motion W_1 to the event $\{W_1 = W_2\}$ is the distribution of Brownian motion with diffusion rate $\frac{1}{2}$

Goal: Find the conditional distribution

$$\mathbb{P}\left\{X \in \cdot | X \in \mathbf{Coal}\right\},$$

where $\mathbf{Coal}$ is the set of coalescing paths and

1. $X(t) = (W_1(t), \ldots, W_n(t))$, $t \geq 0$, and W_k are independent Brownian motions;
2. $X(u, t) = u + W_t(u)$, $u \in [0, 1]$, $t \geq 0$, and W is a cylindrical Wiener process in $L_2 := L_2[0, 1]$.

Simple observation

Observation: Let W_1, W_2 be independent standard Brownian motions on $\mathbb{R}$.

$$\mathbb{P}\{W_1 \in A | W_1 = W_2\} = \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A \middle| \frac{W_1 - W_2}{2} = 0\right\}$$
$$= \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A\right\}$$

The conditional distribution of the standard Brownian motion W_1 to the event $\{W_1 = W_2\}$ is the distribution of Brownian motion with diffusion rate $\frac{1}{2}$

Goal: Find the conditional distribution

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\},$$

where $\mathbf{Coal}$ is the set of coalescing paths and

1. $X(t) = (W_1(t), \ldots, W_n(t))$, $t \geq 0$, and W_k are independent Brownian motions;
2. $X(u, t) = u + W_t(u)$, $u \in [0, 1]$, $t \geq 0$, and W is a cylindrical Wiener process in $L_2 := L_2[0, 1]$.

Simple observation

Observation: Let W_1, W_2 be independent standard Brownian motions on $\mathbb{R}$.

$$\mathbb{P}\left\{W_1 \in A | W_1 = W_2\right\} = \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A \middle| \frac{W_1 - W_2}{2} = 0\right\}$$
$$= \mathbb{P}\left\{\frac{W_1 + W_2}{2} \in A\right\}$$

The conditional distribution of the standard Brownian motion W_1 to the event $\{W_1 = W_2\}$ is the distribution of Brownian motion with diffusion rate $\frac{1}{2}$

Goal: Find the conditional distribution

$$\mathbb{P}\left\{X \in \cdot | X \in \mathbf{Coal}\right\},$$

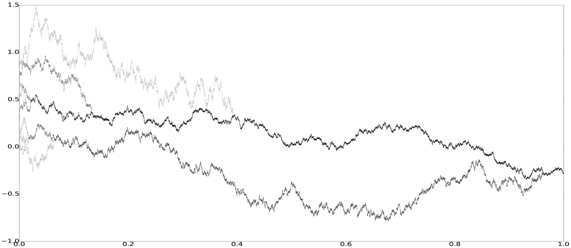
where $\mathbf{Coal}$ is the set of coalescing paths and

1. $X(t) = (W_1(t), \ldots, W_n(t))$, $t \geq 0$, and W_k are independent Brownian motions;
2. $X(u,t) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, and W is a cylindrical Wiener process in $L_2 := L_2[0,1]$.

Our guess

The conditional distribution of a family of independent Brownian motions to the event of coalescing paths is the **modified massive Arratia flow:**

1. particles move independently and coalesce after meeting;
2. each particle has a mass that obeys the conservation law;
3. diffusion rate of each particle is inversely proportional to its mass.

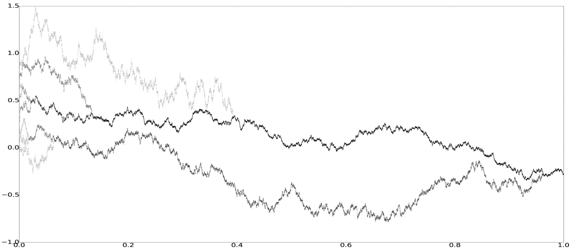


We will justify our guess for finite and infinite dimensional cases. However, the infinite dimensional case will be much more complicated.

Our guess

The conditional distribution of a family of independent Brownian motions to the event of coalescing paths is the **modified massive Arratia flow:**

1. particles move independently and coalesce after meeting;
2. each particle has a mass that obeys the conservation law;
3. diffusion rate of each particle is inversely proportional to its mass.



We will justify our guess for finite and infinite dimensional cases. However, the infinite dimensional case will be much more complicated.

Definition of conditional probability

Let $\mathbf{E}$ be a Polish space, X be a random element in $\mathbf{E}$ and $C \subset \mathbf{E}$.

How can we define $\mathbb{P}\{X \in \cdot | X \in C\}$ if $\mathbb{P}\{X \in C\} = 0$?

Let $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ satisfying $\mathrm{T}^{-1}(\{z_0\}) = C$. Then we will define

$$\mathbb{P}\{X \in \cdot | X \in C\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\} := p(\cdot, z_0),$$

where p is the regular conditional probability of X given $\mathrm{T}(X)$, i.e

1. for every $z \in \mathbf{F}$, $p(\cdot, z)$ is probab. measure on $\mathbf{E}$;
2. for every $A \in \mathcal{B}(\mathbf{E})$, $z \mapsto p(A, z)$ is measurable;
3. for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$,

$$\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} = \int_B p(A, z)\, \mathbb{P}^{\mathrm{T}(X)}(\mathrm{d}z).$$

Remark: If p' is other regular conditional probability of X given $T(X)$, then

$$p'(\cdot, z) = p(\cdot, z)$$

for $\mathbb{P}^{\mathrm{T}(X)}$-a.a. z.

If $z \mapsto p(\cdot, z)$ is continuous at z_0, then $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$ is well-defined.

Definition of conditional probability

Let $\mathbf{E}$ be a Polish space, X be a random element in $\mathbf{E}$ and $C \subset \mathbf{E}$.

How can we define $\mathbb{P}\{X \in \cdot | X \in C\}$ if $\mathbb{P}\{X \in C\} = 0$?

Let $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ satisfying $\mathrm{T}^{-1}(\{z_0\}) = C$. Then we will define

$$\mathbb{P}\{X \in \cdot | X \in C\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\} := p(\cdot, z_0),$$

where p is the regular conditional probability of X given $\mathrm{T}(X)$, i.e

1. for every $z \in \mathbf{F}$, $p(\cdot, z)$ is probab. measure on $\mathbf{E}$;
2. for every $A \in \mathcal{B}(\mathbf{E})$, $z \mapsto p(A, z)$ is measurable;
3. for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$,

$$\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} = \int_B p(A, z)\, \mathbb{P}^{\mathrm{T}(X)}(\mathrm{d}z).$$

Remark: If p' is other regular conditional probability of X given $T(X)$, then

$$p'(\cdot, z) = p(\cdot, z)$$

for $\mathbb{P}^{\mathrm{T}(X)}$-a.a. z.

If $z \mapsto p(\cdot, z)$ is continuous at z_0, then $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$ is well-defined.

Definition of conditional probability

Let $\mathbf{E}$ be a Polish space, X be a random element in $\mathbf{E}$ and $C \subset \mathbf{E}$.

How can we define $\mathbb{P}\{X \in \cdot | X \in C\}$ if $\mathbb{P}\{X \in C\} = 0$?

Let $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ satisfying $\mathrm{T}^{-1}(\{z_0\}) = C$. Then we will define

$$\mathbb{P}\{X \in \cdot | X \in C\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\} := p(\cdot, z_0),$$

where p is the regular conditional probability of X given $\mathrm{T}(X)$, i.e

1. for every $z \in \mathbf{F}$, $p(\cdot, z)$ is probab. measure on $\mathbf{E}$;
2. for every $A \in \mathcal{B}(\mathbf{E})$, $z \mapsto p(A, z)$ is measurable;
3. for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$,

$$\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} = \int_B p(A, z)\, \mathbb{P}^{\mathrm{T}(X)}(\mathrm{d}z).$$

Remark: If p' is other regular conditional probability of X given $T(X)$, then

$$p'(\cdot, z) = p(\cdot, z)$$

for $\mathbb{P}^{\mathrm{T}(X)}$-a.a. z.

If $z \mapsto p(\cdot, z)$ is continuous at z_0, then $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$ is well-defined.

Definition of conditional probability

Let $\mathbf{E}$ be a Polish space, X be a random element in $\mathbf{E}$ and $C \subset \mathbf{E}$.

How can we define $\mathbb{P}\{X \in \cdot | X \in C\}$ if $\mathbb{P}\{X \in C\} = 0$?

Let $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ satisfying $\mathrm{T}^{-1}(\{z_0\}) = C$. Then we will define

$$\mathbb{P}\{X \in \cdot | X \in C\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\} := p(\cdot, z_0),$$

where p is the regular conditional probability of X given $\mathrm{T}(X)$, i.e

1. for every $z \in \mathbf{F}$, $p(\cdot, z)$ is probab. measure on $\mathbf{E}$;
2. for every $A \in \mathcal{B}(\mathbf{E})$, $z \mapsto p(A, z)$ is measurable;
3. for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$,

$$\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} = \int_B p(A, z)\, \mathbb{P}^{\mathrm{T}(X)}(\mathrm{d}z).$$

Remark: If p' is other regular conditional probability of X given $T(X)$, then

$$p'(\cdot, z) = p(\cdot, z)$$

for $\mathbb{P}^{\mathrm{T}(X)}$-a.a. z.

If $z \mapsto p(\cdot, z)$ is continuous at z_0, then $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$ is well-defined.

Definition of conditional probability

Let $\mathbf{E}$ be a Polish space, X be a random element in $\mathbf{E}$ and $C \subset \mathbf{E}$.

How can we define $\mathbb{P}\{X \in \cdot | X \in C\}$ if $\mathbb{P}\{X \in C\} = 0$?

Let $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ satisfying $\mathrm{T}^{-1}(\{z_0\}) = C$. Then we will define

$$\mathbb{P}\{X \in \cdot | X \in C\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\} := p(\cdot, z_0),$$

where p is the regular conditional probability of X given $\mathrm{T}(X)$, i.e

1. for every $z \in \mathbf{F}$, $p(\cdot, z)$ is probab. measure on $\mathbf{E}$;
2. for every $A \in \mathcal{B}(\mathbf{E})$, $z \mapsto p(A, z)$ is measurable;
3. for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$,

$$\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} = \int_B p(A, z)\, \mathbb{P}^{\mathrm{T}(X)}(\mathrm{d}z).$$

Remark: If p' is other regular conditional probability of X given $T(X)$, then

$$p'(\cdot, z) = p(\cdot, z)$$

for $\mathbb{P}^{\mathrm{T}(X)}$-a.a. z.

If $z \mapsto p(\cdot, z)$ is continuous at z_0, then $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$ is well-defined.

Main result: Finite dimensional case

Theorem. (K., Marx '20)

Let $X = (W_1, \dots, W_n)$, where W_k are independent Brownian motions with diffusion rates σ_k^2 (with masses $m_k = \frac{1}{\sigma_k^2}$; assume: $m_1 + \dots + m_n = 1$) starting from $x_1^0 < \dots < x_n^0$, and

$$\mathbf{Coal} = \left\{ (x_k)_{k=1}^n \in C[0,\infty)^n : \begin{array}{l} \forall k, l \in [n],\ \forall s \geq 0,\ x_k(s) = x_l(s) \\ \text{implies } x_k(t) = x_l(t),\ \forall t \geq s \end{array} \right\}.$$

Then $\exists\ \mathrm{T} : C[0,\infty)^n \to C_0[0,\infty)^{n-1}$ such that $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$ and

$$\mathbb{P}\left\{X \in \cdot | X \in \mathbf{Coal}\right\} = \mathbb{P}\left\{X \in \cdot | \mathrm{T}(X) = 0\right\}$$

is the distribution of the modified massive Arratia flow $Y = (Y_1, \dots, Y_n)$, that is,

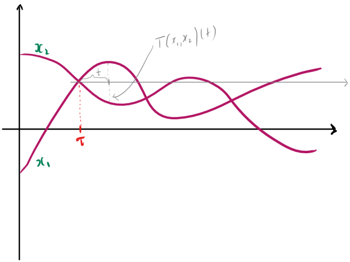
1. Y_k are continuous square-integrable martingales;
2. $Y_k(0) = x_k^0$;
3. for $k < l$, $Y_k(t) \leq Y_l(t)$;
4. $\langle Y_k, Y_l \rangle_t = \int_0^t \frac{\mathbb{I}_{\{Y_k(s) = Y_l(s)\}}}{m_k(s)} ds$, where $m_k(t) = \sum_{l \in [n]:\ Y_k(t) = Y_l(t)} m_l$;

Example: Two particle system

Let $m_1 = m_2 = \frac{1}{2}$. Then

$$\mathrm{T}(x_1, x_2)(t) = \begin{cases} \frac{x_2(\tau+t) - x_1(\tau+t)}{2}, & \text{if } \tau < \infty, \\ 0, & \text{if } \tau = \infty, \end{cases} \quad t \geq 0,$$

where $\tau = \inf\{t \geq 0 : \ x_1(t) = x_2(t)\}$.



and $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$.

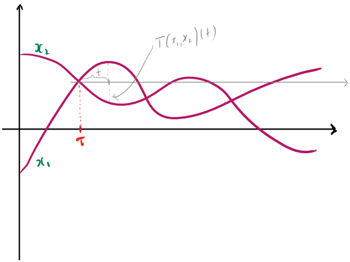
In general, $\mathrm{T}(x_1, \ldots, x_n)$ defines "the difference between coordinate functions x_k after their meeting"

Example: Two particle system

Let $m_1 = m_2 = \frac{1}{2}$. Then

$$\mathrm{T}(x_1, x_2)(t) = \begin{cases} \frac{x_2(\tau+t) - x_1(\tau+t)}{2}, & \text{if } \tau < \infty, \\ 0, & \text{if } \tau = \infty, \end{cases} \qquad t \geq 0,$$

where $\tau = \inf\{t \geq 0 : \ x_1(t) = x_2(t)\}$.



and $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$.

In general, $\mathrm{T}(x_1, \ldots, x_n)$ defines "the difference between coordinate functions x_k after their meeting"

Construction of regular conditional probability

Let X be a random element in $\mathbf{E}$ and $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ is a measurable map.

Assume that there exists a quadruple $(\mathbf{G}, \Psi, Y, Z)$ satisfying

(P1) $\mathbf{G}$ is a measurable space;

(P2) Y and Z are independent random elements in $\mathbf{G}$ and $\mathbf{F}$, respectively;

(P3) $\Psi : \mathbf{G} \times \mathbf{F} \to \mathbf{E}$ is measurable and $X \overset{d}{=} \Psi(Y, Z)$;

(P4) $\mathrm{T}(\Psi(Y, Z)) = Z$ a.s.

Proposition

Let $(\mathbf{G}, \Psi, Y, Z)$ satisfy (P1)-(P4). Then

$$\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z\} = p(\cdot, z) = \mathbb{P}\{\Psi(Y, z) \in \cdot\}, \quad z \in \mathbf{F},$$

is a regular conditional probability of X given $\mathrm{T}(X)$.

Proof. $\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} \overset{(P3)}{=} \mathbb{P}\{\Psi(Y, Z) \in A,\ \mathrm{T}(\Psi(Y, Z)) \in B\}$

$\overset{(P4)}{=} \mathbb{P}\{\Psi(Y, Z) \in A,\ Z \in B\} \overset{(P2)}{=} \int_B \mathbb{P}\{\Psi(Y, z) \in A\} \mathbb{P}^Z(dz) \overset{(P4)}{=} \int_B \mathbb{P}\{\Psi(Y, z) \in A\} \mathbb{P}^{T(X)}(dz)$

Construction of regular conditional probability

Let X be a random element in $\mathbf{E}$ and $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ is a measurable map.

Assume that there exists a quadruple $(\mathbf{G}, \Psi, Y, Z)$ satisfying

(P1) $\mathbf{G}$ is a measurable space;

(P2) Y and Z are independent random elements in $\mathbf{G}$ and $\mathbf{F}$, respectively;

(P3) $\Psi : \mathbf{G} \times \mathbf{F} \to \mathbf{E}$ is measurable and $X \stackrel{d}{=} \Psi(Y, Z)$;

(P4) $\mathrm{T}(\Psi(Y, Z)) = Z$ a.s.

Proposition

Let $(\mathbf{G}, \Psi, Y, Z)$ satisfy (P1)-(P4). Then

$$\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z\} = p(\cdot, z) = \mathbb{P}\{\Psi(Y, z) \in \cdot\}, \quad z \in \mathbf{F},$$

is a regular conditional probability of X given $\mathrm{T}(X)$.

Proof. $\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} \stackrel{(P3)}{=} \mathbb{P}\{\Psi(Y, Z) \in A,\ \mathrm{T}(\Psi(Y, Z)) \in B\}$

$$\stackrel{(P4)}{=} \mathbb{P}\{\Psi(Y, Z) \in A,\ Z \in B\} \stackrel{(P2)}{=} \int_B \mathbb{P}\{\Psi(Y, z) \in A\} \mathbb{P}^Z(dz) \stackrel{(P4)}{=} \int_B \mathbb{P}\{\Psi(Y, z) \in A\} \mathbb{P}^{T(X)}(dz)$$

Construction of regular conditional probability

Let X be a random element in $\mathbf{E}$ and $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ is a measurable map.

Assume that there exists a quadruple $(\mathbf{G}, \Psi, Y, Z)$ satisfying

(P1) $\mathbf{G}$ is a measurable space;

(P2) Y and Z are independent random elements in $\mathbf{G}$ and $\mathbf{F}$, respectively;

(P3) $\Psi : \mathbf{G} \times \mathbf{F} \to \mathbf{E}$ is measurable and $X \overset{d}{=} \Psi(Y, Z)$;

(P4) $\mathrm{T}(\Psi(Y, Z)) = Z$ a.s.

Proposition

Let $(\mathbf{G}, \Psi, Y, Z)$ satisfy (P1)-(P4). Then

$$\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z\} = p(\cdot, z) = \mathbb{P}\{\Psi(Y, z) \in \cdot\}, \quad z \in \mathbf{F},$$

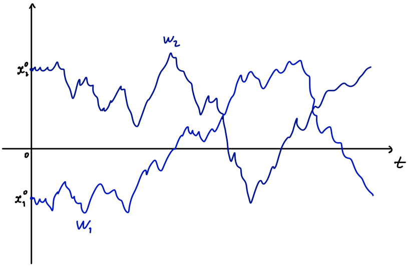
is a regular conditional probability of X given $\mathrm{T}(X)$.

Proof. $\mathbb{P}\{X \in A,\ \mathrm{T}(X) \in B\} \overset{(P3)}{=} \mathbb{P}\{\Psi(Y, Z) \in A,\ \mathrm{T}(\Psi(Y, Z)) \in B\}$

$\overset{(P4)}{=} \mathbb{P}\{\Psi(Y, Z) \in A,\ Z \in B\} \overset{(P2)}{=} \int_B \mathbb{P}\{\Psi(Y, z) \in A\} \mathbb{P}^Z(dz) \overset{(P4)}{=} \int_B \mathbb{P}\{\Psi(Y, z) \in A\} \mathbb{P}^{T(X)}(dz)$

Conditional probability: Two particle case

$X = (W_1, W_2)$, where W_k are indep. BM with diff rates $\frac{1}{m_k} = 2$ starting from x_k^0.



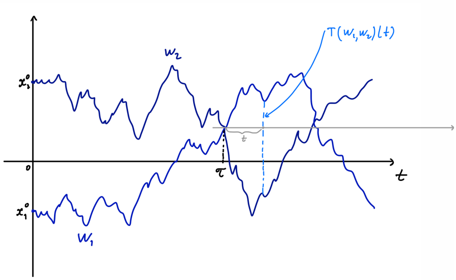
$$\mathbf{E} = C[0,\infty)^2, \quad \mathbf{F} = C_0[0,\infty) \quad \text{and} \quad \mathrm{T} : \mathbf{E} \to \mathbf{F}$$

$$\mathrm{T}(X)(t) = \mathrm{T}(W_1, W_2)(t) = \begin{cases} \frac{W_2(\tau+t) - W_1(\tau+t)}{2}, & \tau < \infty, \\ 0, & \tau = \infty, \end{cases}$$

where $\tau = \inf\{t \geq 0 : \ W_1(t) = W_2(t)\}$.

Conditional probability: Two particle case

$X = (W_1, W_2)$, where W_k are indep. BM with diff rates $\frac{1}{m_k} = 2$ starting from x_k^0.



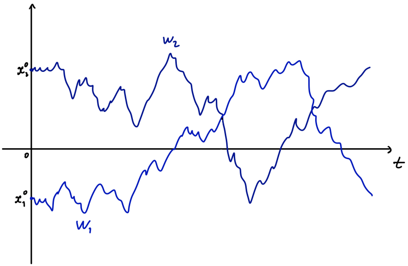
$$\mathbf{E} = C[0,\infty)^2, \quad \mathbf{F} = C_0[0,\infty) \quad \text{and} \quad \mathrm{T} : \mathbf{E} \to \mathbf{F}$$

$$\mathrm{T}(X)(t) = \mathrm{T}(W_1, W_2)(t) = \begin{cases} \frac{W_2(\tau+t) - W_1(\tau+t)}{2}, & \tau < \infty, \\ 0, & \tau = \infty, \end{cases}$$

where $\tau = \inf\{t \geq 0 : \; W_1(t) = W_2(t)\}$.

Two particle system: Regular conditional distribution

$$X \overset{d}{=} \Psi(Y,Z); \quad Y \perp\!\!\!\perp Z; \quad \mathrm{T}(\Psi(Y,Z)) = Z \text{ a.s.} \implies p(\cdot, z) = \mathrm{Law}\,\Psi(Y,z)$$



1. Coalescing part Y of X is a strong solution to the equation

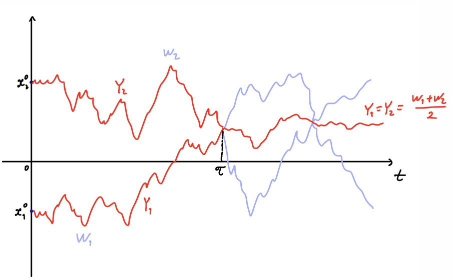
$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

2. $\Psi(Y,Z)(t) = \big(Y_1(t) - Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}, Y_2(t) + Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}\big)$,

   Z is a standard BM indep. of Y

Two particle system: Regular conditional distribution

$$X \overset{d}{=} \Psi(Y,Z); \quad Y \perp\!\!\!\perp Z; \quad \mathrm{T}(\Psi(Y,Z)) = Z \text{ a.s.} \implies p(\cdot, z) = \mathrm{Law}\,\Psi(Y,z)$$



1. Coalescing part Y of X is a strong solution to the equation

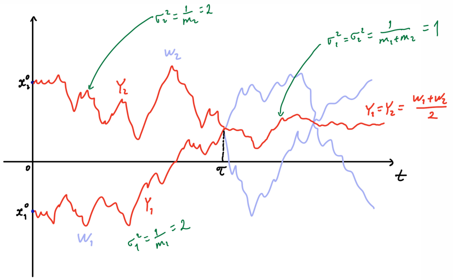
$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t\ge\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t\ge\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

2. $\Psi(Y,Z)(t) = \left(Y_1(t) - Z(t-\tau)\mathbb{I}_{\{t\ge\tau\}}, Y_2(t) + Z(t-\tau)\mathbb{I}_{\{t\ge\tau\}}\right)$,

   Z is a standard BM indep. of Y

Two particle system: Regular conditional distribution

$$X \overset{d}{=} \Psi(Y,Z); \quad Y \perp\!\!\!\perp Z; \quad \mathrm{T}(\Psi(Y,Z)) = Z \text{ a.s.} \quad \Longrightarrow \quad p(\cdot, z) = \mathrm{Law}\,\Psi(Y,z)$$



1. Coalescing part Y of X is a strong solution to the equation

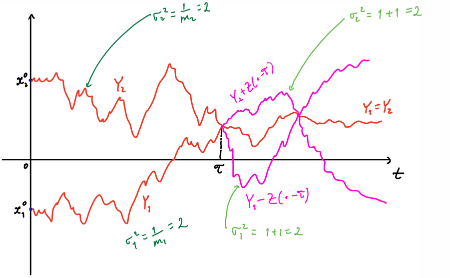
$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

2. $\Psi(Y,Z)(t) = \left(Y_1(t) - Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}, Y_2(t) + Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}\right)$,

   Z is a standard BM indep. of Y

Two particle system: Regular conditional distribution

$$X \overset{d}{=} \Psi(Y,Z); \quad Y \perp\!\!\!\perp Z; \quad \mathrm{T}(\Psi(Y,Z)) = Z \text{ a.s.} \implies p(\cdot, z) = \mathrm{Law}\,\Psi(Y,z)$$



1. Coalescing part Y of X is a strong solution to the equation

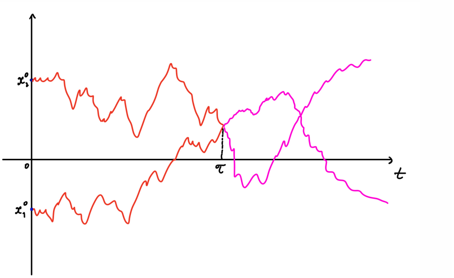
$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t \geq \tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t \geq \tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

2. $\Psi(Y,Z)(t) = \big(Y_1(t) - Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}, Y_2(t) + Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}\big)$,

   Z is a standard BM indep. of Y

Two particle system: Regular conditional distribution

$X \overset{d}{=} \Psi(Y,Z)$; $\quad Y \perp\!\!\!\perp Z$; $\quad \mathrm{T}(\Psi(Y,Z)) = Z$ a.s. $\implies p(\cdot, z) = \mathrm{Law}\,\Psi(Y,z)$



1. Coalescing part Y of X is a strong solution to the equation

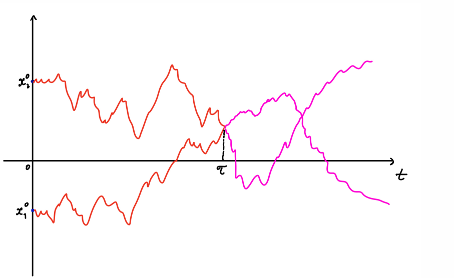
$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

2. $\Psi(Y,Z)(t) = \big(Y_1(t) - Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}, Y_2(t) + Z(t-\tau)\mathbb{I}_{\{t\geq\tau\}}\big)$,

   Z is a standard BM indep. of Y

Two particle system: Regular conditional distribution

$X \overset{d}{=} \Psi(Y, Z)$; $\quad Y \perp\!\!\!\perp Z$; $\quad \mathrm{T}(\Psi(Y, Z)) = Z$ a.s. $\implies p(\cdot, z) = \mathrm{Law}\, \Psi(Y, z)$



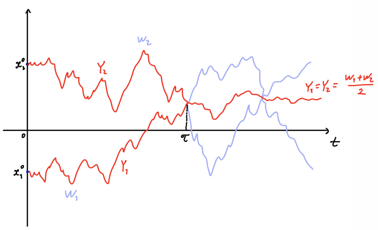
1. Coalescing part Y of X is a strong solution to the equation

$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t\geq\tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

2. $\mathrm{T}(\Psi(Y, Z))(t) = \frac{1}{2}\Big(Y_2(\tau + t - \tau) + Z(\tau + t - \tau) - Y_1(\tau + t - \tau) + Z(\tau + t - \tau)\Big)$
$= Z(t)$ a.s. (for $\tau < +\infty$)

Two particle system: Continuity

$$p(\cdot, z) = \mathbb{P}\left\{X \in \cdot | T(X) = z\right\} = \mathbb{P}\left\{\Psi(Y, z) \in \cdot\right\} \quad \text{for } \mathbb{P}^Z\text{-a.a. } z$$



Since $z \mapsto \Psi(Y, z) = \left(Y_1(t) - z(t - \tau^Y)\mathbb{I}_{\{t \geq \tau^Y\}}, Y_2(t) + z(t - \tau^Y)\mathbb{I}_{\{t \geq \tau^Y\}}\right)$ is continuous,

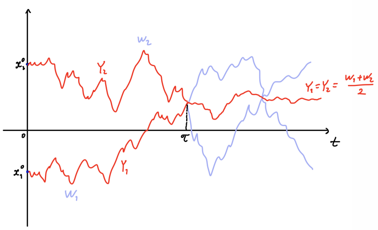
$$\mathbb{P}\left\{X \in \cdot | X \in \mathbf{Coal}\right\} = \mathbb{P}\left\{X \in \cdot | T(X) = 0\right\} = \mathbb{P}\left\{Y \in \cdot\right\},$$

where Y is the coalescing part of X:

$$\begin{cases} dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t \geq \tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t \geq \tau\}} d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0) = x_1^0, \quad Y_2(0) = x_2^0, \end{cases}$$

Two particle system: Continuity

$$p(\cdot,z)=\mathbb{P}\{X\in\cdot|T(X)=z\}=\mathbb{P}\{\Psi(Y,z)\in\cdot\}\quad \text{for } \mathbb{P}^Z\text{-a.a. } z$$



Since $z\mapsto\Psi(Y,z)=\big(Y_1(t)-z(t-\tau^Y)\mathbb{I}_{\{t\ge\tau^Y\}},Y_2(t)+z(t-\tau^Y)\mathbb{I}_{\{t\ge\tau^Y\}}\big)$ is continuous,

$$\mathbb{P}\{X\in\cdot|X\in\mathbf{Coal}\}=\mathbb{P}\{X\in\cdot|T(X)=0\}=\mathbb{P}\{Y\in\cdot\},$$

where Y is the coalescing part of X:

$$\begin{cases} dY_1(t)=\mathbb{I}_{\{t<\tau\}}dW_1(t)+\mathbb{I}_{\{t\ge\tau\}}d\frac{W_1(t)+W_2(t)}{2}, \\ dY_2(t)=\mathbb{I}_{\{t<\tau\}}dW_2(t)+\mathbb{I}_{\{t\ge\tau\}}d\frac{W_1(t)+W_2(t)}{2}, \\ Y_1(0)=x_1^0,\quad Y_2(0)=x_2^0, \end{cases}$$

Finite number of particles: Coalescing part

Let $X = W = (W_1, \dots, W_n)$, be independent independent Brownian particles with masses m_k, $m_1 + \dots + m_n = 1$, starting from $x_1^0 < \dots < x_n^0$.

Inner product on $\mathbb{R}^n$: $\langle x, y\rangle_m = \sum_{k=1}^n x_k y_k m_k$ and denote pr_x^m the orthogonal projection onto $\mathbb{R}^n(x) := \{y : \ y_k = y_l \text{ if } x_k = x_l\}$.
$\langle W, a\rangle_m$ is a Brownian motion with diffusion rate $\|a\|_m^2$

Coalescing part of X:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}_{Y(s)}^m \, dW(s), \quad Y_1(t) \le \dots \le Y_n(t), \quad t \ge 0$$

The equation has a unique strong solution

Lemma (Splitting part of X)

Let B be an independent copy of W. Then

$$\tilde{W}(t) := Y(t) + \int_0^t (\mathrm{pr}_{Y(s)}^m)^\perp dB(s)$$

has the same distribution as W.

$\langle \tilde{W}, a\rangle_m$ is a continuous martingale with quadratic variation $\left[\langle \tilde{W}, a\rangle_m\right]_t = \int_0^t \|\mathrm{pr}_{Y(s)}^m a\|^2 ds + \int_0^t \|(\mathrm{pr}_{Y(s)}^m)^\perp a\|^2 ds = \|a\|^2 t.$

Finite number of particles: Coalescing part

Let $X = W = (W_1, \ldots, W_n)$, be independent independent Brownian particles with masses m_k, $m_1 + \cdots + m_n = 1$, starting from $x_1^0 < \cdots < x_n^0$.

Inner product on $\mathbb{R}^n$: $\langle x, y \rangle_m = \sum_{k=1}^n x_k y_k m_k$ and denote pr_x^m the orthogonal projection onto $\mathbb{R}^n(x) := \{y : \ y_k = y_l \text{ if } x_k = x_l\}$.

$\langle W, a \rangle_m$ is a Brownian motion with diffusion rate $\|a\|_m^2$

Coalescing part of X:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}_{Y(s)}^m \, dW(s), \quad Y_1(t) \leq \cdots \leq Y_n(t), \quad t \geq 0$$

The equation has a unique strong solution

Lemma (Splitting part of X)

Let B be an independent copy of W. Then

$$\tilde{W}(t) := Y(t) + \int_0^t (\mathrm{pr}_{Y(s)}^m)^\perp dB(s)$$

has the same distribution as W.

$\langle \tilde{W}, a \rangle_m$ is a continuous martingale with quadratic variation $\left[\langle \tilde{W}, a \rangle_m\right]_t = \int_0^t \| \mathrm{pr}_{Y(s)}^m \, a\|^2 ds + \int_0^t \|(\mathrm{pr}_{Y(s)}^m)^\perp a\|^2 ds = \|a\|^2 t.$

Finite number of particles: Coalescing part

Let $X = W = (W_1, \dots, W_n)$, be independent independent Brownian particles with masses m_k, $m_1 + \dots + m_n = 1$, starting from $x_1^0 < \dots < x_n^0$.

Inner product on $\mathbb{R}^n$: $\langle x, y \rangle_m = \sum_{k=1}^n x_k y_k m_k$ and denote pr_x^m the orthogonal projection onto $\mathbb{R}^n(x) := \{y : \ y_k = y_l \text{ if } x_k = x_l\}$.
$\langle W, a \rangle_m$ is a Brownian motion with diffusion rate $\|a\|_m^2$

Coalescing part of X:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}_{Y(s)}^m \, dW(s), \quad Y_1(t) \le \dots \le Y_n(t), \quad t \ge 0$$

The equation has a unique strong solution

Lemma (Splitting part of X)

Let B be an independent copy of W. Then

$$\tilde{W}(t) := Y(t) + \int_0^t (\mathrm{pr}_{Y(s)}^m)^\perp dB(s)$$

has the same distribution as W.

$\langle \tilde{W}, a \rangle_m$ is a continuous martingale with quadratic variation $\left[\langle \tilde{W}, a \rangle_m\right]_t = \int_0^t \|\mathrm{pr}_{Y(s)}^m a\|^2 ds + \int_0^t \|(\mathrm{pr}_{Y(s)}^m)^\perp a\|^2 ds = \|a\|^2 t.$

Finite number of particles: Coalescing part

Let $X = W = (W_1, \dots, W_n)$, be independent independent Brownian particles with masses m_k, $m_1 + \dots + m_n = 1$, starting from $x_1^0 < \dots < x_n^0$.

Inner product on $\mathbb{R}^n$: $\langle x, y\rangle_m = \sum_{k=1}^n x_k y_k m_k$ and denote pr_x^m the orthogonal projection onto $\mathbb{R}^n(x) := \{y : \ y_k = y_l \text{ if } x_k = x_l\}$.
$\langle W, a\rangle_m$ is a Brownian motion with diffusion rate $\|a\|_m^2$

Coalescing part of X:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}_{Y(s)}^m \, dW(s), \quad Y_1(t) \le \dots \le Y_n(t), \quad t \ge 0$$

The equation has a unique strong solution

Lemma (Splitting part of X)

Let B be an independent copy of W. Then

$$\tilde{W}(t) := Y(t) + \int_0^t (\mathrm{pr}_{Y(s)}^m)^{\perp} dB(s)$$

has the same distribution as W.

$\langle \tilde{W}, a\rangle_m$ is a continuous martingale with quadratic variation $\left[\langle \tilde{W}, a\rangle_m\right]_t = \int_0^t \|\mathrm{pr}_{Y(s)}^m a\|^2 ds + \int_0^t \|(\mathrm{pr}_{Y(s)}^m)^{\perp} a\|^2 ds = \|a\|^2 t$.

Finite number of particles: Coalescing part

Let $X = W = (W_1, \dots, W_n)$, be independent independent Brownian particles with masses m_k, $m_1 + \cdots + m_n = 1$, starting from $x_1^0 < \cdots < x_n^0$.

Inner product on $\mathbb{R}^n$: $\langle x, y\rangle_m = \sum_{k=1}^n x_k y_k m_k$ and denote pr_x^m the orthogonal projection onto $\mathbb{R}^n(x) := \{y : \ y_k = y_l \text{ if } x_k = x_l\}$.
$\langle W, a\rangle_m$ is a Brownian motion with diffusion rate $\|a\|_m^2$

Coalescing part of X:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}_{Y(s)}^m \, dW(s), \quad Y_1(t) \le \cdots \le Y_n(t), \quad t \ge 0$$

The equation has a unique strong solution

Lemma (Splitting part of X)

Let B be an independent copy of W. Then

$$\tilde{W}(t) := Y(t) + \int_0^t (\mathrm{pr}_{Y(s)}^m)^{\perp} dB(s)$$

has the same distribution as W.

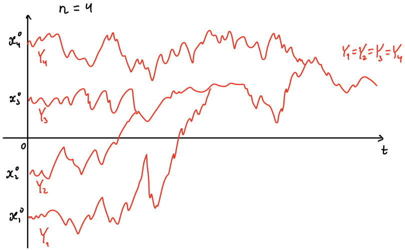
$\langle \tilde{W}, a\rangle_m$ is a continuous martingale with quadratic variation $\left[\langle \tilde{W}, a\rangle_m\right]_t = \int_0^t \| \mathrm{pr}_{Y(s)}^m a\|^2 ds + \int_0^t \|(\mathrm{pr}_{Y(s)}^m)^{\perp} a\|^2 ds = \|a\|^2 t.$

Basis generated by coalescing part

Let Y be the coalescing part of W:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}^m_{Y(s)}\, dW(s), \quad Y_1(t) \le \cdots \le Y_n(t), \quad t \ge 0.$$

Define the stopping times τ_k^Y and basis e_k^Y, $k = 0, \ldots, n-1$ as follows:



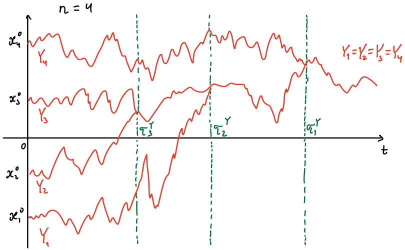
e_k^Y is a unit vector in $\mathbb{R}^n(Y(\tau_{k+1}^Y)) \ominus \mathbb{R}^n(Y(\tau_k^Y))$.

Basis generated by coalescing part

Let Y be the coalescing part of W:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}^m_{Y(s)}\, dW(s), \quad Y_1(t) \le \cdots \le Y_n(t), \quad t \ge 0.$$

Define the stopping times τ_k^Y and basis e_k^Y, $k = 0, \ldots, n-1$ as follows:



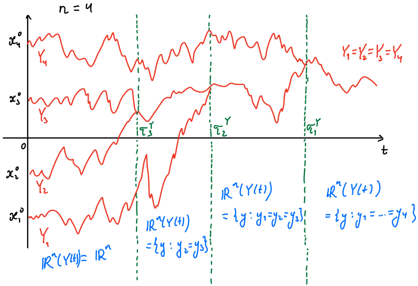
e_k^Y is a unit vector in $\mathbb{R}^n(Y(\tau_{k+1}^Y)) \ominus \mathbb{R}^n(Y(\tau_k^Y))$.

Basis generated by coalescing part

Let Y be the coalescing part of W:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}^m_{Y(s)}\, dW(s), \quad Y_1(t) \le \cdots \le Y_n(t), \quad t \ge 0.$$

Define the stopping times τ_k^Y and basis e_k^Y, $k = 0, \ldots, n-1$ as follows:



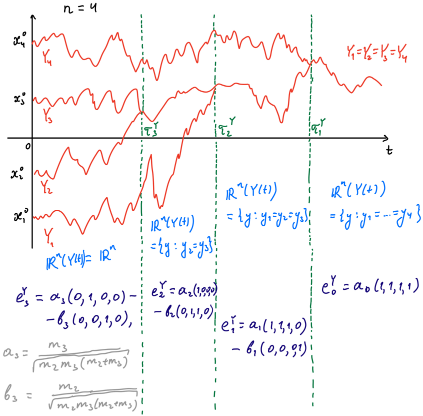
e_k^Y is a unit vector in $\mathbb{R}^n(Y(\tau_{k+1}^Y)) \ominus \mathbb{R}^n(Y(\tau_k^Y))$.

Basis generated by coalescing part

Let Y be the coalescing part of W:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}^m_{Y(s)}\, dW(s), \quad Y_1(t) \le \cdots \le Y_n(t), \quad t \ge 0.$$

Define the stopping times τ_k^Y and basis e_k^Y, $k = 0, \dots, n-1$ as follows:



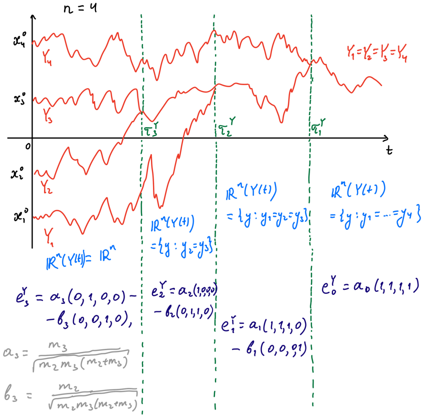
e_k^Y is a unit vector in $\mathbb{R}^n(Y(\tau_{k+1}^Y)) \ominus \mathbb{R}^n(Y(\tau_k^Y))$.

Basis generated by coalescing part

Let Y be the coalescing part of W:

$$Y(t) = x^0 + \int_0^t \mathrm{pr}^m_{Y(s)}\, dW(s), \quad Y_1(t) \le \cdots \le Y_n(t), \quad t \ge 0.$$

Define the stopping times τ_k^Y and basis e_k^Y, $k = 0, \ldots, n-1$ as follows:



e_k^Y is a unit vector in $\mathbb{R}^n(Y(\tau_{k+1}^Y)) \ominus \mathbb{R}^n(Y(\tau_k^Y))$.

Maps Ψ and T

Map Ψ:

$$\tilde{W}(t) = Y(t) + \int_0^t (\mathrm{pr}^m_{Y(s)})^\perp dB(s) = Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y \left(\langle B(t), e_k^Y \rangle_m - \langle B(\tau_k^Y), e_k^Y \rangle_m \right)$$

$$\stackrel{d}{=} Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

for Z_k, $k = [n-1]$, standard independent BM independent of Y.

Map T:

$$\mathrm{T}(W)(t) := \left(\langle W(t + \tau_l^Y), e_l^Y \rangle_m \right)_{l=1,\dots,n-1}$$

Conditional distribution:

Then $X = W \stackrel{d}{=} \Psi(Y, Z)$, $T(\Psi(Y, Z)) = Z$ a.s., $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$,
$z \mapsto \Psi(Y, z)$ is continuous.

Hence,

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = 0\} = \mathbb{P}\{Y \in \cdot\}$$

Maps Ψ and T

Map Ψ:

$$\tilde{W}(t) = Y(t) + \int_0^t (\mathrm{pr}^m_{Y(s)})^{\perp} dB(s) = Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y \left(\langle B(t), e_k^Y \rangle_m - \langle B(\tau_k^Y), e_k^Y \rangle_m \right)$$

$$\stackrel{d}{=} Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

for Z_k, $k = [n-1]$, standard independent BM independent of Y.

Map T:

$$\mathrm{T}(W)(t) := \left(\langle W(t + \tau_l^Y), e_l^Y \rangle_m \right)_{l=1,\dots,n-1}$$

Conditional distribution:

Then $X = W \stackrel{d}{=} \Psi(Y, Z)$, $T(\Psi(Y, Z)) = Z$ a.s., $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$,
$z \mapsto \Psi(Y, z)$ is continuous.

Hence,

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = 0\} = \mathbb{P}\{Y \in \cdot\}$$

Maps Ψ and T

Map Ψ:

$$\tilde{W}(t) = Y(t) + \int_0^t (\mathrm{pr}^m_{Y(s)})^{\perp} dB(s) = Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y \left(\langle B(t), e_k^Y \rangle_m - \langle B(\tau_k^Y), e_k^Y \rangle_m \right)$$

$$\stackrel{d}{=} Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y) =: \Psi(Y, Z)$$

for Z_k, $k = [n-1]$, standard independent BM independent of Y.

Map T:

$$\mathrm{T}(W)(t) := \left(\langle W(t + \tau_l^Y), e_l^Y \rangle_m \right)_{l=1,\dots,n-1}$$

Conditional distribution:

Then $X = W \stackrel{d}{=} \Psi(Y, Z)$, $T(\Psi(Y, Z)) = Z$ a.s., $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$,
$z \mapsto \Psi(Y, z)$ is continuous.

Hence,

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = 0\} = \mathbb{P}\{Y \in \cdot\}$$

Maps Ψ and T

Map Ψ:

$$\tilde{W}(t) = Y(t) + \int_0^t (\mathrm{pr}^m_{Y(s)})^\perp dB(s) = Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y \left(\langle B(t), e_k^Y \rangle_m - \langle B(\tau_k^Y), e_k^Y \rangle_m \right)$$

$$\stackrel{d}{=} Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y) =: \Psi(Y, Z)$$

for Z_k, $k = [n-1]$, standard independent BM independent of Y.

Map T:

$$\mathrm{T}(W)(t) := \left(\langle W(t + \tau_l^Y), e_l^Y \rangle_m \right)_{l=1,\dots,n-1}$$

Conditional distribution:

Then $X = W \stackrel{d}{=} \Psi(Y, Z)$, $T(\Psi(Y, Z)) = Z$ a.s., $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$,
$z \mapsto \Psi(Y, z)$ is continuous.

Hence,

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = 0\} = \mathbb{P}\{Y \in \cdot\}$$

Maps Ψ and T

Map Ψ:

$$\tilde{W}(t) = Y(t) + \int_0^t (\mathrm{pr}^m_{Y(s)})^{\perp} dB(s) = Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y \left(\langle B(t), e_k^Y \rangle_m - \langle B(\tau_k^Y), e_k^Y \rangle_m \right)$$

$$\stackrel{d}{=} Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y) =: \Psi(Y, Z)$$

for Z_k, $k = [n-1]$, standard independent BM independent of Y.

Map T:

$$\mathrm{T}(W)(t) := \left(\langle W(t + \tau_l^Y), e_l^Y \rangle_m \right)_{l=1,\dots,n-1}$$

Conditional distribution:

Then $X = W \stackrel{d}{=} \Psi(Y, Z)$, $T(\Psi(Y, Z)) = Z$ a.s., $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$, $z \mapsto \Psi(Y, z)$ is continuous.

Hence,

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = 0\} = \mathbb{P}\{Y \in \cdot\}$$

Maps Ψ and T

Map Ψ:

$$\tilde{W}(t) = Y(t) + \int_0^t (\mathrm{pr}^m_{Y(s)})^{\perp} dB(s) = Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y \left(\langle B(t), e_k^Y \rangle_m - \langle B(\tau_k^Y), e_k^Y \rangle_m \right)$$

$$\stackrel{d}{=} Y(t) + \sum_{k=1}^{n-1} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y) =: \Psi(Y, Z)$$

for Z_k, $k = [n-1]$, standard independent BM independent of Y.

Map T:

$$\mathrm{T}(W)(t) := \left(\langle W(t + \tau_l^Y), e_l^Y \rangle_m \right)_{l=1,\dots,n-1}$$

Conditional distribution:

Then $X = W \stackrel{d}{=} \Psi(Y, Z)$, $T(\Psi(Y, Z)) = Z$ a.s., $\mathrm{T}^{-1}(\{0\}) = \mathbf{Coal}$,
$z \mapsto \Psi(Y, z)$ is continuous.

Hence,

$$\mathbb{P}\{X \in \cdot | X \in \mathbf{Coal}\} = \mathbb{P}\{X \in \cdot | \mathrm{T}(X) = 0\} = \mathbb{P}\{Y \in \cdot\}$$

Infinite particle system

$$X_t(u) = u + W_t(u), \quad u \in [0,1], \quad t \geq 0,$$

is a cylindrical Wiener process in $L_2 = L_2[0,1]$.

Disintegration of cylindrical Wiener process

Let $X_t(u) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, be a cylindrical Wiener process in L_2.

Coalescing part:

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow} := \{g \in L_2 : \ g \uparrow\}, \tag{1}$$

where $\mathrm{id}(u) = u$, $u \in [0,1]$, and pr_g is the orthogonal projection onto subspace of $\sigma(g)$-measurable functions in L_2.

Map Ψ:

$$\Psi(Y,Z)(t) = Y(t) + \sum_{k=1}^{\infty} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

Construction problems:

1. X does not take values in L_2;
2. Equation (1) admits weak solutions, not necessarily unique;
3. $z \mapsto \Psi(Y,z)$ is not continuous, so we cannot take a value of the regular conditional probability at a fixed point.

Disintegration of cylindrical Wiener process

Let $X_t(u) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, be a cylindrical Wiener process in L_2.

Coalescing part:

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow} := \{g \in L_2 :\ g \uparrow\}, \tag{1}$$

where $\mathrm{id}(u) = u$, $u \in [0,1]$, and pr_g is the orthogonal projection onto subspace of $\sigma(g)$-measurable functions in L_2.

Map Ψ:

$$\Psi(Y,Z)(t) = Y(t) + \sum_{k=1}^{\infty} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

Construction problems:

1. X does not take values in L_2;
2. Equation (1) admits weak solutions, not necessarily unique;
3. $z \mapsto \Psi(Y,z)$ is not continuous, so we cannot take a value of the regular conditional probability at a fixed point.

Disintegration of cylindrical Wiener process

Let $X_t(u) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, be a cylindrical Wiener process in L_2.

Coalescing part:

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow} := \{g \in L_2 : \ g \uparrow\}, \tag{1}$$

where $\mathrm{id}(u) = u$, $u \in [0,1]$, and pr_g is the orthogonal projection onto subspace of $\sigma(g)$-measurable functions in L_2.

Map Ψ:

$$\Psi(Y,Z)(t) = Y(t) + \sum_{k=1}^{\infty} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

Construction problems:

1. X does not take values in L_2;
2. Equation (1) admits weak solutions, not necessarily unique;
3. $z \mapsto \Psi(Y,z)$ is not continuous, so we cannot take a value of the regular conditional probability at a fixed point.

Disintegration of cylindrical Wiener process

Let $X_t(u) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, be a cylindrical Wiener process in L_2.

Coalescing part:

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow} := \{g \in L_2 : \ g \uparrow\}, \tag{1}$$

where $\mathrm{id}(u) = u$, $u \in [0,1]$, and pr_g is the orthogonal projection onto subspace of $\sigma(g)$-measurable functions in L_2.

Map Ψ:

$$\Psi(Y, Z)(t) = Y(t) + \sum_{k=1}^{\infty} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

Construction problems:

1. X does not take values in L_2;
2. Equation (1) admits weak solutions, not necessarily unique;
3. $z \mapsto \Psi(Y, z)$ is not continuous, so we cannot take a value of the regular conditional probability at a fixed point.

Disintegration of cylindrical Wiener process

Let $X_t(u) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, be a cylindrical Wiener process in L_2.

Coalescing part:

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow} := \{g \in L_2 : \ g \uparrow\}, \tag{1}$$

where $\mathrm{id}(u) = u$, $u \in [0,1]$, and pr_g is the orthogonal projection onto subspace of $\sigma(g)$-measurable functions in L_2.

Map Ψ:

$$\Psi(Y,Z)(t) = Y(t) + \sum_{k=1}^{\infty} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

Construction problems:

1. X does not take values in L_2;
2. Equation (1) admits weak solutions, not necessarily unique;
3. $z \mapsto \Psi(Y,z)$ is not continuous, so we cannot take a value of the regular conditional probability at a fixed point.

Disintegration of cylindrical Wiener process

Let $X_t(u) = u + W_t(u)$, $u \in [0,1]$, $t \geq 0$, be a cylindrical Wiener process in L_2.

Coalescing part:

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow} := \{g \in L_2 : \ g \uparrow\}, \tag{1}$$

where $\mathrm{id}(u) = u$, $u \in [0,1]$, and pr_g is the orthogonal projection onto subspace of $\sigma(g)$-measurable functions in L_2.

Map Ψ:

$$\Psi(Y,Z)(t) = Y(t) + \sum_{k=1}^{\infty} \mathbb{I}_{\{t \geq \tau_k^Y\}} e_k^Y Z_k(t - \tau_k^Y)$$

Construction problems:

1. X does not take values in L_2;
2. Equation (1) admits weak solutions, not necessarily unique;
3. $z \mapsto \Psi(Y,z)$ is not continuous, so we cannot take a value of the regular conditional probability at a fixed point.

Value of regular conditional probability along a sequence

Let X be a random element in a Polish space $\mathbf{E}$ and $T : \mathbf{E} \to \mathbf{F}$.

Let p be a regular conditional probability of X given $\mathrm{T}(X)$.

Consider random elements $\{\xi^n,\ n \geq 1\}$ in a metric space $\mathbf{F}$ such that

(B1) $\mathbb{P}^{\xi^n} \ll \mathbb{P}^{\mathrm{T}(X)}$ for all $n \geq 1$,

(B2) $\xi^n \xrightarrow{d} z_0$ in $\mathbf{F}$.

A probability measure ν on $\mathbf{E}$ is the value at z_0 of the regular conditional probability p along $\{\xi^n\}$ if for every $f \in C_b(\mathbf{E})$

$$\mathbb{E}\int_{\mathbf{E}} f(x)p(dx,\xi^n) \to \int_{\mathbf{E}} f(x)\nu(dx), \quad n \to \infty.$$

The measure ν represents the conditional distribution $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$.

Lemma

Let $z_0 \in \operatorname{supp} \mathbb{P}^{T(X)}$. $\exists\ \nu$ which is the value at z_0 of p along any sequence $\{\xi^n\}$ iff p has a version continuous at z_0. In this case, ν is its value at z_0.

Value of regular conditional probability along a sequence

Let X be a random element in a Polish space $\mathbf{E}$ and $T : \mathbf{E} \to \mathbf{F}$.

Let p be a regular conditional probability of X given $\mathrm{T}(X)$.

Consider random elements $\{\xi^n,\ n \geq 1\}$ in a metric space $\mathbf{F}$ such that

(B1) $\mathbb{P}^{\xi^n} \ll \mathbb{P}^{\mathrm{T}(X)}$ for all $n \geq 1$,

(B2) $\xi^n \xrightarrow{d} z_0$ in $\mathbf{F}$.

A probability measure ν on $\mathbf{E}$ is the value at z_0 of the regular conditional probability p along $\{\xi^n\}$ if for every $f \in C_b(\mathbf{E})$

$$\mathbb{E}\int_{\mathbf{E}} f(x)p(dx,\xi^n) \to \int_{\mathbf{E}} f(x)\nu(dx), \quad n \to \infty.$$

The measure ν represents the conditional distribution $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$.

Lemma

Let $z_0 \in \operatorname{supp} \mathbb{P}^{T(X)}$. $\exists\ \nu$ which is the value at z_0 of p along any sequence $\{\xi^n\}$ iff p has a version continuous at z_0. In this case, ν is its value at z_0.

Value of regular conditional probability along a sequence

Let X be a random element in a Polish space $\mathbf{E}$ and $T : \mathbf{E} \to \mathbf{F}$.

Let p be a regular conditional probability of X given $\mathrm{T}(X)$.

Consider random elements $\{\xi^n,\ n \geq 1\}$ in a metric space $\mathbf{F}$ such that

(B1) $\mathbb{P}^{\xi^n} \ll \mathbb{P}^{\mathrm{T}(X)}$ for all $n \geq 1$,

(B2) $\xi^n \xrightarrow{d} z_0$ in $\mathbf{F}$.

A probability measure ν on $\mathbf{E}$ is the value at z_0 of the regular conditional probability p along $\{\xi^n\}$ if for every $f \in C_b(\mathbf{E})$

$$\mathbb{E}\int_{\mathbf{E}} f(x)p(dx, \xi^n) \to \int_{\mathbf{E}} f(x)\nu(dx), \quad n \to \infty.$$

The measure ν represents the conditional distribution $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$.

Lemma

Let $z_0 \in \operatorname{supp} \mathbb{P}^{T(X)}$. $\exists\ \nu$ which is the value at z_0 of p along any sequence $\{\xi^n\}$ iff p has a version continuous at z_0. In this case, ν is its value at z_0.

Value of regular conditional probability along a sequence

Let X be a random element in a Polish space $\mathbf{E}$ and $T : \mathbf{E} \to \mathbf{F}$.

Let p be a regular conditional probability of X given $\mathrm{T}(X)$.

Consider random elements $\{\xi^n, \ n \geq 1\}$ in a metric space $\mathbf{F}$ such that

(B1) $\mathbb{P}^{\xi^n} \ll \mathbb{P}^{\mathrm{T}(X)}$ for all $n \geq 1$,

(B2) $\xi^n \xrightarrow{d} z_0$ in $\mathbf{F}$.

A probability measure ν on $\mathbf{E}$ is the value at z_0 of the regular conditional probability p along $\{\xi^n\}$ if for every $f \in C_b(\mathbf{E})$

$$\mathbb{E} \int_{\mathbf{E}} f(x) p(dx, \xi^n) \to \int_{\mathbf{E}} f(x) \nu(dx), \quad n \to \infty.$$

The measure ν represents the conditional distribution $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$.

Lemma

Let $z_0 \in \operatorname{supp} \mathbb{P}^{T(X)}$. $\exists\ \nu$ which is the value at z_0 of p along any sequence $\{\xi^n\}$ iff p has a version continuous at z_0. In this case, ν is its value at z_0.

Value of regular conditional probability along a sequence

Let X be a random element in a Polish space $\mathbf{E}$ and $T : \mathbf{E} \to \mathbf{F}$.

Let p be a regular conditional probability of X given $\mathrm{T}(X)$.

Consider random elements $\{\xi^n,\ n \geq 1\}$ in a metric space $\mathbf{F}$ such that

(B1) $\mathbb{P}^{\xi^n} \ll \mathbb{P}^{\mathrm{T}(X)}$ for all $n \geq 1$,

(B2) $\xi^n \xrightarrow{d} z_0$ in $\mathbf{F}$.

A probability measure ν on $\mathbf{E}$ is the value at z_0 of the regular conditional probability p along $\{\xi^n\}$ if for every $f \in C_b(\mathbf{E})$

$$\mathbb{E} \int_{\mathbf{E}} f(x) p(dx, \xi^n) \to \int_{\mathbf{E}} f(x) \nu(dx), \quad n \to \infty.$$

The measure ν represents the conditional distribution $\mathbb{P}\{X \in \cdot | \mathrm{T}(X) = z_0\}$.

Lemma

Let $z_0 \in \operatorname{supp} \mathbb{P}^{T(X)}$. $\exists\ \nu$ which is the value at z_0 of p along any sequence $\{\xi^n\}$ iff p has a version continuous at z_0. In this case, ν is its value at z_0.

Approximation and value along a sequence

Let C be a closed set in $\mathbf{E}$. One usually defines

$$\mathbb{P}\{X \in \cdot\, | X \in C\} = \lim_{\varepsilon \to 0} \mathbb{P}\{X \in \cdot\, | X \in C_\varepsilon\},$$

where $C_\varepsilon = \{x \in \mathbf{E} :\ d_{\mathbf{E}}(C, x) < \varepsilon\}$.

Take

$$\mathrm{T}(x) := d_{\mathbf{E}}(C, x), \quad x \in \mathbf{E},$$

and note that $\mathrm{T}^{-1}(\{0\}) = C$.

Set $\xi := T(X)$ and define random elements ξ^ε by

$$\mathbb{P}\{\xi^\varepsilon \in A\} = \frac{1}{\mathbb{P}\{\xi < \varepsilon\}} \int_A \mathbb{I}_{\{x<\varepsilon\}} \mathbb{P}^\xi(dx) = \mathbb{P}\{\xi \in A | X \in C_\varepsilon\}, \quad A \in \mathcal{B}(\mathbf{E}).$$

Then $\{\xi^\varepsilon,\ \varepsilon > 0\}$, satisfies (B1), (B2) with $z_0 = 0$ and

$$\mathbb{E}\int_{\mathbf{E}} f(x) p(dx, \xi^\varepsilon) = \frac{\mathbb{E} f(X) \mathbb{I}_{\{\xi<\varepsilon\}}}{\mathbb{P}\{\xi < \varepsilon\}} = \frac{\mathbb{E} f(X) \mathbb{I}_{\{X \in C_\varepsilon\}}}{\mathbb{P}\{X \in C_\varepsilon\}}$$

$$= \int_{\mathbf{E}} f(x) \mathbb{P}\{X \in dx | X \in C_\varepsilon\},$$

where p is the regular conditional probability of X given ξ.

Approximation and value along a sequence

Let C be a closed set in $\mathbf{E}$. One usually defines

$$\mathbb{P}\{X \in \cdot \,|X \in C\} = \lim_{\varepsilon \to 0} \mathbb{P}\{X \in \cdot \,|X \in C_\varepsilon\},$$

where $C_\varepsilon = \{x \in \mathbf{E} : \; d_{\mathbf{E}}(C, x) < \varepsilon\}$.

Take

$$\mathrm{T}(x) := d_{\mathbf{E}}(C, x), \quad x \in \mathbf{E},$$

and note that $\mathrm{T}^{-1}(\{0\}) = C$.

Set $\xi := T(X)$ and define random elements ξ^ε by

$$\mathbb{P}\{\xi^\varepsilon \in A\} = \frac{1}{\mathbb{P}\{\xi < \varepsilon\}} \int_A \mathbb{I}_{\{x<\varepsilon\}} \mathbb{P}^\xi(dx) = \mathbb{P}\{\xi \in A | X \in C_\varepsilon\}, \quad A \in \mathcal{B}(\mathbf{E}).$$

Then $\{\xi^\varepsilon, \; \varepsilon > 0\}$, satisfies (B1), (B2) with $z_0 = 0$ and

$$\mathbb{E}\int_{\mathbf{E}} f(x) p(dx, \xi^\varepsilon) = \frac{\mathbb{E} f(X) \mathbb{I}_{\{\xi<\varepsilon\}}}{\mathbb{P}\{\xi < \varepsilon\}} = \frac{\mathbb{E} f(X) \mathbb{I}_{\{X \in C_\varepsilon\}}}{\mathbb{P}\{X \in C_\varepsilon\}}$$
$$= \int_{\mathbf{E}} f(x) \mathbb{P}\{X \in dx | X \in C_\varepsilon\},$$

where p is the regular conditional probability of X given ξ.

Approximation and value along a sequence

Let C be a closed set in $\mathbf{E}$. One usually defines

$$\mathbb{P}\left\{X \in \cdot \,|X \in C\right\} = \lim_{\varepsilon \to 0} \mathbb{P}\left\{X \in \cdot \,|X \in C_\varepsilon\right\},$$

where $C_\varepsilon = \{x \in \mathbf{E} : \; d_{\mathbf{E}}(C, x) < \varepsilon\}$.

Take

$$\mathrm{T}(x) := d_{\mathbf{E}}(C, x), \quad x \in \mathbf{E},$$

and note that $\mathrm{T}^{-1}(\{0\}) = C$.

Set $\xi := T(X)$ and define random elements ξ^ε by

$$\mathbb{P}\left\{\xi^\varepsilon \in A\right\} = \frac{1}{\mathbb{P}\left\{\xi < \varepsilon\right\}} \int_A \mathbb{I}_{\{x<\varepsilon\}} \mathbb{P}^\xi(dx) = \mathbb{P}\left\{\xi \in A | X \in C_\varepsilon\right\}, \quad A \in \mathcal{B}(\mathbf{E}).$$

Then $\{\xi^\varepsilon, \; \varepsilon > 0\}$, satisfies (B1), (B2) with $z_0 = 0$ and

$$\mathbb{E} \int_{\mathbf{E}} f(x) p(dx, \xi^\varepsilon) = \frac{\mathbb{E} f(X) \mathbb{I}_{\{\xi<\varepsilon\}}}{\mathbb{P}\left\{\xi < \varepsilon\right\}} = \frac{\mathbb{E} f(X) \mathbb{I}_{\{X \in C_\varepsilon\}}}{\mathbb{P}\left\{X \in C_\varepsilon\right\}}$$

$$= \int_{\mathbf{E}} f(x) \mathbb{P}\left\{X \in dx | X \in C_\varepsilon\right\},$$

where p is the regular conditional probability of X given ξ.

Approximation and value along a sequence

Let C be a closed set in $\mathbf{E}$. One usually defines

$$\mathbb{P}\left\{X \in \cdot\,|X \in C\right\} = \lim_{\varepsilon \to 0} \mathbb{P}\left\{X \in \cdot\,|X \in C_\varepsilon\right\},$$

where $C_\varepsilon = \{x \in \mathbf{E} :\ d_{\mathbf{E}}(C, x) < \varepsilon\}$.

Take

$$\mathrm{T}(x) := d_{\mathbf{E}}(C, x), \quad x \in \mathbf{E},$$

and note that $\mathrm{T}^{-1}(\{0\}) = C$.

Set $\xi := T(X)$ and define random elements ξ^ε by

$$\mathbb{P}\left\{\xi^\varepsilon \in A\right\} = \frac{1}{\mathbb{P}\left\{\xi < \varepsilon\right\}} \int_A \mathbb{I}_{\{x<\varepsilon\}} \mathbb{P}^\xi(dx) = \mathbb{P}\left\{\xi \in A|X \in C_\varepsilon\right\}, \quad A \in \mathcal{B}(\mathbf{E}).$$

Then $\{\xi^\varepsilon,\ \varepsilon > 0\}$, satisfies (B1), (B2) with $z_0 = 0$ and

$$\mathbb{E}\int_{\mathbf{E}} f(x)p(dx, \xi^\varepsilon) = \frac{\mathbb{E}f(X)\mathbb{I}_{\{\xi<\varepsilon\}}}{\mathbb{P}\left\{\xi < \varepsilon\right\}} = \frac{\mathbb{E}f(X)\mathbb{I}_{\{X\in C_\varepsilon\}}}{\mathbb{P}\left\{X \in C_\varepsilon\right\}}$$
$$= \int_{\mathbf{E}} f(x)\mathbb{P}\left\{X \in dx|X \in C_\varepsilon\right\},$$

where p is the regular conditional probability of X given ξ.

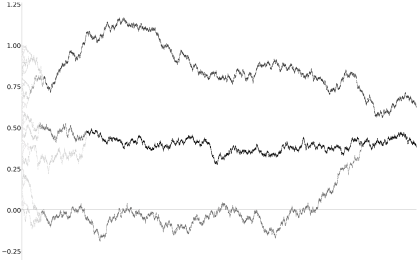
Coalescing part and state space

There exists a continuous process Y in L_2 and a cylindrical Wiener process W such that

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow}, \quad t \geq 0.$$

The process Y has a modification $\{Y_t(u),\ u \in [0,1],\ t \geq 0\}$ such that

1. $Y(u)$ is a continuous square-integrable martingale;
2. $Y_0(u) = u$;
3. for $u < v$, $Y_t(u) \leq Y_t(v)$;
4. $\langle Y(u), Y(v) \rangle_t = \int_0^t \frac{\mathbb{I}_{\{Y_s(u)=Y_s(v)\}}}{m(u,s)} ds$, where $m(u,s) = \mathrm{Leb}\,\{v :\ Y_s(v) = Y_s(u)\}$.



Take $X := (Y, W)$ and find the conditional distrib. to the event of coal. paths for W.

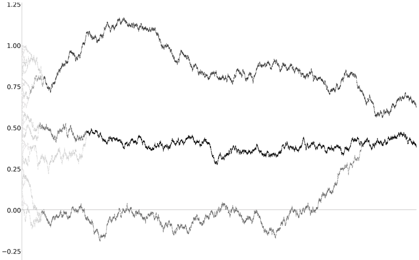
Coalescing part and state space

There exists a continuous process Y in L_2 and a cylindrical Wiener process W such that

$$Y_t = \mathrm{id} + \int_0^t \mathrm{pr}_{Y_s}\, dW_s, \quad Y_t \in L_2^{\uparrow}, \quad t \geq 0.$$

The process Y has a modification $\{Y_t(u),\ u \in [0,1],\ t \geq 0\}$ such that

1. $Y(u)$ is a continuous square-integrable martingale;
2. $Y_0(u) = u$;
3. for $u < v$, $Y_t(u) \leq Y_t(v)$;
4. $\langle Y(u), Y(v)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{Y_s(u)=Y_s(v)\}}}{m(u,s)} ds$, where $m(u,s) = \mathrm{Leb}\,\{v : \ Y_s(v) = Y_s(u)\}$.



Take $X := (Y, W)$ and find the conditional distrib. to the event of coal. paths for W.

State space

Let $h_j,\ j \geq 0$, be a fixed orthonormal basis in L_2 with $h_0 = 1$.

Set $\mathbf{E} := C\left([0,\infty), L_2\right) \times C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$ and identify

$$W = \sum_{j=0}^{\infty} h_j \langle W, h_j \rangle \quad \longleftrightarrow \quad \widehat{W} = \left(\langle W, h_j \rangle\right)_{j \geq 0} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$$

Define as before $\cdots < \tau_n < \cdots < \tau_1 < \infty$ and $e_k^Y,\ k \geq 0$.

$$\mathrm{T}_t(X) = \mathrm{T}_t(Y, W) = \sum_{k=1}^{\infty} e_k^Y \langle W_{t+\tau_k^Y}, e_k^Y \rangle, \quad t \geq 0,$$

is cylindrical Wiener process in $L_2^0 = L_2 \ominus \{\text{constant functions}\}$.

$$\Psi_t(Y, Z) = \left(Y_t, Y_t + \sum_{k=1}^{\infty} e_k^Y \mathbb{I}_{\{t \geq \tau_k^Y\}} \langle Z_{t-\tau_k^Y}, e_k^Y \rangle \right), \quad t \geq 0,$$

where Z is a cylindrical Wiener process in L_2^0, identified with

$$Z = \sum_{j=1}^{\infty} h_j \langle Z, h_j \rangle \quad \longleftrightarrow \quad \widehat{Z} = \left(\langle Z, h_j \rangle\right)_{j \geq 1} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{N}} =: \mathbf{F}.$$

State space

Let $h_j, \ j \geq 0$, be a fixed orthonormal basis in L_2 with $h_0 = 1$.

Set $\mathbf{E} := C\left([0,\infty), L_2\right) \times C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$ and identify

$$W = \sum_{j=0}^{\infty} h_j \langle W, h_j \rangle \quad \longleftrightarrow \quad \widehat{W} = \left(\langle W, h_j \rangle\right)_{j \geq 0} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$$

Define as before $\cdots < \tau_n < \cdots < \tau_1 < \infty$ and $e_k^Y, \ k \geq 0$.

$$\mathrm{T}_t(X) = \mathrm{T}_t(Y, W) = \sum_{k=1}^{\infty} e_k^Y \langle W_{t+\tau_k^Y}, e_k^Y \rangle, \quad t \geq 0,$$

is cylindrical Wiener process in $L_2^0 = L_2 \ominus \{\text{constant functions}\}$.

$$\Psi_t(Y, Z) = \left(Y_t, Y_t + \sum_{k=1}^{\infty} e_k^Y \mathbb{I}_{\{t \geq \tau_k^Y\}} \langle Z_{t-\tau_k^Y}, e_k^Y \rangle \right), \quad t \geq 0,$$

where Z is a cylindrical Wiener process in L_2^0, identified with

$$Z = \sum_{j=1}^{\infty} h_j \langle Z, h_j \rangle \quad \longleftrightarrow \quad \widehat{Z} = \left(\langle Z, h_j \rangle\right)_{j \geq 1} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{N}} =: \mathbf{F}.$$

State space

Let $h_j,\ j \geq 0$, be a fixed orthonormal basis in L_2 with $h_0 = 1$.

Set $\mathbf{E} := C\left([0,\infty), L_2\right) \times C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$ and identify

$$W = \sum_{j=0}^{\infty} h_j \langle W, h_j \rangle \quad \longleftrightarrow \quad \widehat{W} = \left(\langle W, h_j \rangle\right)_{j \geq 0} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$$

Define as before $\cdots < \tau_n < \cdots < \tau_1 < \infty$ and $e_k^Y,\ k \geq 0$.

$$\mathrm{T}_t(X) = \mathrm{T}_t(Y, W) = \sum_{k=1}^{\infty} e_k^Y \langle W_{t+\tau_k^Y}, e_k^Y \rangle, \quad t \geq 0,$$

is cylindrical Wiener process in $L_2^0 = L_2 \ominus \{\text{constant functions}\}$.

$$\Psi_t(Y, Z) = \left(Y_t, Y_t + \sum_{k=1}^{\infty} e_k^Y \mathbb{I}_{\{t \geq \tau_k^Y\}} \langle Z_{t-\tau_k^Y}, e_k^Y \rangle \right), \quad t \geq 0,$$

where Z is a cylindrical Wiener process in L_2^0, identified with

$$Z = \sum_{j=1}^{\infty} h_j \langle Z, h_j \rangle \quad \longleftrightarrow \quad \widehat{Z} = \left(\langle Z, h_j \rangle\right)_{j \geq 1} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{N}} =: \mathbf{F}.$$

State space

Let $h_j,\ j \geq 0$, be a fixed orthonormal basis in L_2 with $h_0 = 1$.

Set $\mathbf{E} := C\left([0,\infty), L_2\right) \times C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$ and identify

$$W = \sum_{j=0}^{\infty} h_j \langle W, h_j\rangle \quad \longleftrightarrow \quad \widehat{W} = \left(\langle W, h_j\rangle\right)_{j\geq 0} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$$

Define as before $\cdots < \tau_n < \cdots < \tau_1 < \infty$ and $e_k^Y,\ k \geq 0$.

$$\mathrm{T}_t(X) = \mathrm{T}_t(Y, W) = \sum_{k=1}^{\infty} e_k^Y \langle W_{t+\tau_k^Y}, e_k^Y\rangle, \quad t \geq 0,$$

is cylindrical Wiener process in $L_2^0 = L_2 \ominus \{\text{constant functions}\}$.

$$\Psi_t(Y, Z) = \left(Y_t, Y_t + \sum_{k=1}^{\infty} e_k^Y \mathbb{I}_{\{t \geq \tau_k^Y\}} \langle Z_{t-\tau_k^Y}, e_k^Y\rangle\right), \quad t \geq 0,$$

where Z is a cylindrical Wiener process in L_2^0, identified with

$$Z = \sum_{j=1}^{\infty} h_j \langle Z, h_j\rangle \quad \longleftrightarrow \quad \widehat{Z} = \left(\langle Z, h_j\rangle\right)_{j\geq 1} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{N}} =: \mathbf{F}.$$

State space

Let $h_j,\ j \geq 0$, be a fixed orthonormal basis in L_2 with $h_0 = 1$.

Set $\mathbf{E} := C\left([0,\infty), L_2\right) \times C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$ and identify

$$W = \sum_{j=0}^{\infty} h_j \langle W, h_j \rangle \quad \longleftrightarrow \quad \widehat{W} = \left(\langle W, h_j \rangle\right)_{j\geq 0} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{Z}_+}$$

Define as before $\cdots < \tau_n < \cdots < \tau_1 < \infty$ and $e_k^Y,\ k \geq 0$.

$$\mathrm{T}_t(X) = \mathrm{T}_t(Y, W) = \sum_{k=1}^{\infty} e_k^Y \langle W_{t+\tau_k^Y}, e_k^Y \rangle, \quad t \geq 0,$$

is cylindrical Wiener process in $L_2^0 = L_2 \ominus \{\text{constant functions}\}$.

$$\Psi_t(Y, Z) = \left(Y_t, Y_t + \sum_{k=1}^{\infty} e_k^Y \mathbb{I}_{\{t \geq \tau_k^Y\}} \langle Z_{t-\tau_k^Y}, e_k^Y \rangle \right), \quad t \geq 0,$$

where Z is a cylindrical Wiener process in L_2^0, identified with

$$Z = \sum_{j=1}^{\infty} h_j \langle Z, h_j \rangle \quad \longleftrightarrow \quad \widehat{Z} = \left(\langle Z, h_j \rangle\right)_{j\geq 1} \in C\left([0,\infty), \mathbb{R}\right)^{\mathbb{N}} =: \mathbf{F}.$$

Main result: Infinite dimensional case

Theorem (K., Marx)

The law of $(Y, \widehat{Y})$ is the value at 0 of the regular conditional probability of X given $\xi := T(X)$ along the sequence $\{\xi^n\}_{n\geq 1}$, where

$$\begin{cases} d\xi_j^n(t) = -\alpha_j^n \mathbb{I}_{\{t\leq n\}}\xi_j^n(t)dt + d\widehat{\xi}_j(t), \\ \xi_j^n(0) = 0, \end{cases}$$

where $\{\alpha_j^n,\ n, j \geq 1\}$ is a family of non-negative real numbers such that

(O1) for every $n \geq 1$ the series $\sum_{j=1}^{\infty}(\alpha_j^n)^2 < +\infty$;

(O2) for every $j \geq 1$, $\alpha_j^n \to +\infty$ as $n \to \infty$.

References

V. Konarovskyi, V. Marx.
Conditional Distribution of Independent Brownian Motions to Event of Coalescing Paths
arXiv:arXiv:2008.02568.

Thank you!