# On the existence and uniqueness of solutions to the Dean-Kawasaki equation

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joint work with Max von Renesse and Tobias Lehmann

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# Dean-Kawasaki equation

**The Dean-Kawasaki equation:**

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right) + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right) \qquad (\text{DK}_F^\alpha \text{ eq})$$

- $\mu_t$ is a continuous measure-valued process in $\mathbb{R}^d$;
- $\frac{\delta F(\nu)}{\delta\nu}(x) = \lim_{\varepsilon\to 0+}\frac{F(\nu+\varepsilon\delta_x)-F(\nu)}{\varepsilon} = \frac{\partial}{\partial\varepsilon}F(\nu+\varepsilon\delta_x)\Big|_{\varepsilon=0}$ is the functional derivative of $F$;
- $\dot{W}$ is a space-time white nose;
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The equation is used for modeling of behaviour of huge number of particles in the Langevin dynamics.

(K. Kawasaki '94; D. Dean '96; A. Donev, E. Vanden-Eijnden '14, '15; B. Derrida '16; J. Zimmer '19; B. Gess '19)

$F$ corresponds for the interaction between particles

Open questions until now: What is a notion of solution to the D-K equation? Is the equation well-posed?

Today: We completely answer this question in the case of "smooth" $F$

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# Comparison with equation for Super-Brownian motion

**The Dean-Kawasaki equation without interaction:**

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**Equation for the Super-Brownian motion:**

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \sqrt{\mu_t}\dot{W}_t \qquad (\text{SB eq})$$

Set $\langle\varphi,\nu\rangle = \int_{\mathbb{R}^d}\varphi(x)\nu(dx)$

A continuous process $\mu_t \in \mathcal{M}_f(\mathbb{R}^d)$, $t \geq 0$, is a solution to (SB eq) if, for every $\varphi \in \mathrm{C}_b^2(\mathbb{R}^d)$

$$M_\varphi(t) = \langle\varphi,\mu_t\rangle - \langle\varphi,\mu_0\rangle - \frac{\alpha}{2}\int_0^t \langle\Delta\varphi,\mu_s\rangle ds$$

is a martingale with quadratic variation

$$\int_0^t \langle\varphi^2,\mu_s\rangle ds \quad \left(\text{for D-K equation:} \quad \int_0^t \langle|\nabla\varphi|^2,\mu_s\rangle ds\right).$$

A. Etheridge: An Introduction to Superprocesses
See also works of D. Dawson, E. Perkins, L. Mytnik

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# Definition of solutions to (DK$_F^\alpha$ eq)

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**Definition of (martingale) solution**

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# Trivial solutions to (DK$_F^\alpha$ eq) and the main result

Let $X^i(t)$, $t \geq 0$, $i = 1, \dots, n$, be a solution to

$$dX^i(t) = \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}(X^i(t))dt + \sqrt{n}dw^i(t), \quad i = 1, \dots, n$$

where $\mu_t = \frac{1}{n}\sum_{i=1}^n \delta_{X^i(t)}$ and $w_i$ are standard independent BM

By the Itô formula $\mu_t$, $t \geq 0$, is a solution to

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with $\alpha = n$.

**Theorem** (K., T. Lehmann, M. von Renesse)

Let $\mu_0(\mathbb{R}^d) = 1$, and $F : \mathcal{M}_f(\mathbb{R}^d) \to \mathbb{R}$ be bounded and twice continuously differentiable in $\mu$ and $x$ with bounded derivatives. Then the equation has a (unique) solution iff $\alpha = n$ and $\mu_0 = \frac{1}{n}\sum_{i=1}^n \delta_{x^i}$. Moreover, it is defined as above.

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# Strategy of the proof

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

1. Proof in the case $F = 0$.
2. Girsanov transform

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Existence of solutions to

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla \cdot \left(\sqrt{\mu_t}\dot{W}_t\right)$$

# Basic properties of solutions to (DK$_0^\alpha$ eq)

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

- **The equation preserves the total mass**, i.e $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$.
  Take $\varphi \equiv 1$. Then

$$\mu_t(\mathbb{R}^d) = \langle \mu_t, \varphi\rangle = \langle \mu_0, \varphi\rangle + \int_0^t \langle \Delta\varphi, \mu_s\rangle ds + M_\varphi(t)$$

  where the q.v. $[M_\varphi]_t = \int_0^t \langle |\nabla\varphi|^2, \mu_s\rangle ds = 0$.

- **Laplace duality:**

$$\mathbb{E}e^{-\langle \mu_t, f\rangle} = e^{-\langle \mu_0, v(t)\rangle},$$

  where $v$ is a solution to the Hamilton-Jacobi equation:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\alpha}{2}\Delta v - \frac{1}{2}|\nabla v|^2, \\ v|_{t=0} = f \end{cases}$$

$$d_s e^{-\langle \mu_s, v(t-s)\rangle} = e^{-\langle \mu_s, v(t-s)\rangle}$$

$$\cdot\left[\langle mu_s, \partial_s v(t-s) - \frac{\alpha}{2}\Delta v(t-s) + \frac{1}{2}|\nabla v(t-s)|^2\rangle\right] ds + dM$$

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# Hamilton-Jacobi equation and generating function of $\mu_t(A)$

**H-J equation:**

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**Solution to H-J equation:**

$$V_t f = -\alpha \ln\left(P_t e^{-\frac{1}{\alpha}f}\right)$$

where $u(t) = P_t g$ is the solution to the heat equation:

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**Lemma.**

For $A \subset \mathbb{R}^d$ and $t \geq 0$,

$$\mathbb{E}e^{-r\alpha\mu_t(A)} = \mathbb{E}e^{-\langle \mu_t, r\alpha\mathbb{I}_A\rangle} = e^{-\langle \mu_0, V_t(r\alpha\mathbb{I}_A)\rangle} = e^{\alpha\left\langle \mu_0, \ln\left(1+(e^{-r}-1)P_t\mathbb{I}_A\right)\right\rangle}, \quad r > 0$$

Consequently,

$$\mathbb{E}s^{\alpha\mu_t(A)} = e^{\alpha\langle \mu_0, \ln(1+(s-1)P_t\mathbb{I}_A)\rangle}, \quad s = e^{-r} > 0$$

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# Proof of the theorem, $F = 0$

- $\mathbb{E}s^{\alpha\mu_t(A)} = e^{\alpha\langle\mu_0, \ln(1+(s-1)P_t\mathbb{I}_A)\rangle}$ $\quad t \geq 0, \quad A \subset \mathbb{R}^d$ ;
- Let $A$ is bounded and $t > 0$ $\quad\Longrightarrow\quad$ $P_t\mathbb{I}_A \leq 1 - \delta$, for some $\delta > 0$;
- $s \mapsto e^{\alpha\langle\mu_0, \ln(1+(s-1)P_t\mathbb{I}_A)\rangle}$ is well-defined and inf. diff. in a neighbourhood of 0;

**Lemma.**

Let $\xi$ be a nonnegative random variable on $\mathbb{R}$ and $\forall n \geq 1$

$$\mathbb{E}s^{\xi} = \sum_{k=0}^{n} s^k p_k + o(s^n), \quad s \to 0+ .$$

Then $\xi \in \mathbb{N} \cup \{0\}$ a.s. and $\mathbb{P}\{\xi = k\} = p_k, \quad k \geq 1.$

- $\alpha\mu_t(A) \in \mathbb{N} \cup \{0\}$;
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- $\mathbb{E}s^{\alpha\mu_t(A)} = e^{\alpha\langle\mu_0, \ln(1+(s-1)P_t\mathbb{I}_A)\rangle}$ $\quad t \geq 0, \quad A \subset \mathbb{R}^d$ ;
- Let $A$ is bounded and $t > 0 \quad \Longrightarrow \quad P_t\mathbb{I}_A \leq 1 - \delta$, for some $\delta > 0$;
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**Lemma.**

Let $\xi$ be a nonnegative random variable on $\mathbb{R}$ and $\forall n \geq 1$

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Then $\xi \in \mathbb{N} \cup \{0\}$ a.s. and $\mathbb{P}\{\xi = k\} = p_k, \;\; k \geq 1.$

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Existence of solutions to

$$\frac{\partial}{\partial t}\mu_t = \frac{\alpha}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right) + \nabla\cdot\left(\sqrt{\mu_t}\dot{W}_t\right)$$

# A special form of $F$

For $f \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$ we assume that

$$F(\nu) = f\left(\langle \varphi, \nu \rangle\right), \quad \nu \in \mathcal{M}_f(\mathbb{R}^d).$$

$$\frac{\delta F(\nu)}{\delta \nu}(x) = f'(\langle \varphi, \nu \rangle)\varphi(x)$$

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**Itô formula for D-K equation**

Let $G(\nu) = g(\langle \psi, \nu \rangle)$, $g \in C_b^2(\mathbb{R})$, $\psi \in C_b^2(\mathbb{R}^d)$ and $\mu_t$ be a solution to ($\mathrm{DK}_F^\alpha$ eq).

$$M^G(t) = G(\mu_t) - G(\mu_0) - \int_0^t \left[ \frac{\alpha}{2} \left\langle \Delta \frac{\delta G}{\delta \mu_s}, \mu_s \right\rangle - \left\langle \nabla \frac{\delta G}{\delta \mu_s} \cdot \nabla \frac{\delta F}{\delta \mu_s}, \mu_s \right\rangle \right.$$
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# Girsanov transform and proof of the main theorem

$$F(\nu) = f\left(\langle \varphi, \nu \rangle\right), \quad G(\nu) = g(\langle \psi, \nu \rangle) \quad \nu \in \mathcal{M}_f(\mathbb{R}^d).$$

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$$d\mathbb{P}^G = e^{M^G(t) - \frac{1}{2}[M^G]_t} d\mathbb{P} \quad \text{on} \quad \mathcal{F}_t^\mu.$$

Then $\mu_t$ solves on $(\Omega, \mathcal{F}, \mathbb{P}^G)$

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# Girsanov transform and proof of the main theorem

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# The general case

**Remark**

The same result, in particular the Itô formul, holds for

$$F(\nu) = f(\langle \varphi_1, \nu \rangle, \dots, \langle \varphi_k, \nu \rangle) \quad \text{and} \quad G(\nu) = g(\langle \psi_1, \nu \rangle, \dots, \langle \psi_m, \nu \rangle) \qquad \text{(FG)}$$

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Let $F, G : \mathcal{M}_f(\mathbb{R}^d) \to \mathbb{R}$ bounded twice continuously differentiable functions with bounded derivatives, shortly $F, G \in \mathrm{C}_b^{2,2}(\mathcal{M}_f(\mathbb{R}^d))$.

We need $F_n \to F$, $\nabla \frac{\delta F_n(\nu)}{\delta \nu} \to \nabla \frac{\delta F(\nu)}{\delta \nu}, \dots$, $G_n \to G$, $\nabla \frac{\delta G_n(\nu)}{\delta \nu} \to \nabla \frac{\delta G(\nu)}{\delta \nu}, \dots$
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# Bernstein polynomials on $[0,1]^d$ $(d=1)$

For $g:[0,1]\to\mathbb{R}$ define **Bernstein polynomials**

$$B_n(g)(x)=\sum_{k=0}^{n} g\left(\frac{k}{n}\right) C_n^k x^k(1-x)^{n-k}=\sum_{k=0}^{n} g\left(\frac{k}{n}\right)\varphi_k(x), \quad x\in[0,1], \quad n\geq 1$$

**Lemma**

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(For $d\geq 1$ see e.g. Veretennikov '16)

For $\nu\in\mathcal{M}_f([0,1])$ take

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**Lemma**

- Let $g\in \mathsf{C}^m[0,1]$, then $B_n(g)\to g$ in $\mathsf{C}^m[0,1]$ as $n\to\infty$.
- Moreover, if $g_k\to g\in \mathsf{C}^m[0,1]$, then $B_n(g_k)\to g$ in $\mathsf{C}^m[0,1]$ as $k,n\to\infty$.

(For $d\geq 1$ see e.g. Veretennikov '16)

For $\nu\in\mathcal{M}_f([0,1])$ take

$$\chi_n(\nu) := \sum_{k=1}^{n} \langle\varphi_k,\nu\rangle \delta_{\frac{k}{n}}.$$

**Lemma**

$\chi_n(\nu)\to\nu$ in $\mathcal{M}_f([0,1])$. $\quad \left(\langle g,\chi_n(\nu)\rangle = \left\langle \sum_{k=0}^{n} g\left(\frac{k}{n}\right)\varphi_k,\nu\right\rangle \to \langle g,\nu\rangle\right)$

# Approximation of $F \in \mathrm{C}^{2,2}\left(\mathcal{M}_f([0,1]^d)\right)$ $(d=1)$

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Define for $F$

$$F_n(\nu) := F(\chi_n(\nu)) = u(\langle \varphi_1, \nu \rangle, \ldots, \langle \varphi_n, \nu \rangle).$$

Denote

$$F'(\nu; x) := \frac{\delta F(\nu)}{\delta \nu}(x) = \lim_{\varepsilon \to 0+} \frac{F(\nu + \varepsilon \delta_x) - F(\nu)}{\varepsilon} = \frac{\partial F(\nu + \varepsilon \delta_x)}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

Using $\chi_n(\nu + \varepsilon \delta_x) = \chi_n(\nu) + \varepsilon \sum_{k=0}^{n} \varphi_k(x) \delta_{\frac{k}{n}}$, we have

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Similarly, $\frac{\delta^2 F_n(\nu)}{\delta \nu^2} = B_n \otimes B_n(F''(\chi_n(\nu); \cdot, \cdot)) \to F''(\nu; \cdot, \cdot)$ in $\left(\mathrm{C}^2[0,1]\right)^2$.

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**Theorem (Approximation in $\mathcal{M}_f([0,1])$)**

Let $\mathcal{N}(C) = \{\nu \in \mathcal{M}_f([0,1]) : \ \nu([0,1]) \leq C\}$ and $F \in \mathrm{C}^{2,m}(\mathcal{M}_f([0,1]))$. Define $F_n = F \circ \chi_n$. Then

- Then $F_n \to F$ uniformly on $\mathcal{N}(C)$;
- $\frac{\delta F_n(\nu)}{\delta \nu} \to \frac{\delta F(\nu)}{\delta \nu}$ in $\mathrm{C}^m[0,1]$ uniformly on $\mathcal{N}(C)$;
- $\frac{\delta^2 F_n(\nu)}{\delta \nu^2} \to \frac{\delta^2 F(\nu)}{\delta \nu^2}$ in $(\mathrm{C}^m[0,1])^2$ uniformly on $\mathcal{N}(C)$.

# Approximation of $F \in C^{2,m}(\mathcal{M}_f(\mathbb{R}^d))$ $(d = 1)$

Set

$$\theta_\psi(\nu)(dx) := \psi(x)\nu(dx), \quad \nu \in \mathcal{M}_f(\mathbb{R})$$

If $\operatorname{supp}\psi \subset [a,b]$, then $\theta_\psi : \mathcal{M}_f(R) \to \mathcal{M}_f([a,b])$

**Lemma**

Let $\psi_n$ be a sequence of smooth uniformly (in $x$ and $n$) bounded function on $\mathbb{R}$ such that $\psi_n \to 1$ in $C^m(\mathbb{R})$. Then for all $\nu \in \mathcal{M}_f(\mathbb{R})$

$$F(\theta_{\psi_n}(\nu)) \to F(\nu);$$

$$\frac{\delta F(\theta_{\psi_n}(\nu))}{\delta\nu} \to \frac{\delta F(\nu)}{\delta\nu} \quad \text{in} \quad C^m(\mathbb{R});$$

$$\frac{\delta^2 F(\theta_{\psi_n}(\nu))}{\delta\nu^2} \to \frac{\delta^2 F(\nu)}{\delta\nu^2} \quad \text{in} \quad (C^m(\mathbb{R}))^2;$$

The approximation sequernce for $F \in C^{2,m}(\mathcal{M}_f(\mathbb{R}))$ is

$$F_n(\nu) = F(\chi_n^{[a_n,b_n]}(\theta_{\psi_n}(\nu))), \quad \nu \in \mathcal{M}_f(\mathbb{R}),$$

where $\chi_n^{[a,b]}$ is defined similatly as $\chi_n$ but on $\mathcal{M}_f([a,b])$, $\psi_n \to 1$ in $C^m(\mathbb{R})$, $[a_n,b_n] \uparrow \mathbb{R}$.

# Approximation of $F \in \mathsf{C}^{2,m}\left(\mathcal{M}_f(\mathbb{R}^d)\right)$ $(d=1)$

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# Particle systems with singular interaction. Wasserstein diffusion

**Dean-Kawasaki equation with "non-smooth" interaction:**

$$\frac{\partial}{\partial t}\mu_t = \Gamma(\mu_t) + \nabla \cdot \left(\sqrt{\mu_t}\dot{W}_t\right)$$

**Wasserstein Diffusion on $[0,1]$:** von Renesse, Sturm '09

$$\langle \varphi, \Gamma(\mu_t)\rangle = \beta\langle \Delta\varphi, \mu_t\rangle + \sum_{I \in \mathrm{gaps}(\mu_t)} \left[\frac{\varphi''(I_+) + \varphi''(I_-)}{2} - \frac{\varphi'(I_+) - \varphi'(I_-)}{|I|}\right]$$

It can be obtained as a limit of $\mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{x_k^n(nt)}$

$$dx_k^n = \left(\frac{\beta}{n+1} - 1\right)\left(\frac{1}{x_k^n - x_{k-1}^n} - \frac{1}{x_{k+1}^n - x_k^n}\right)dt + \sqrt{2}dw_k + dl_{k-1}^n - dl_k^n$$

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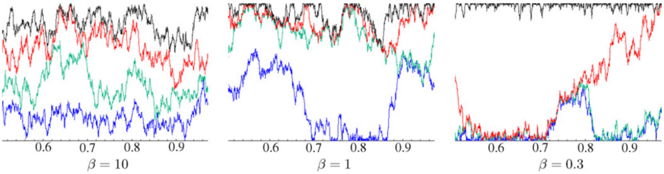

# Particle systems with singular interaction. Modified Arratia flow

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**Modified Arratia flow:** K. '14

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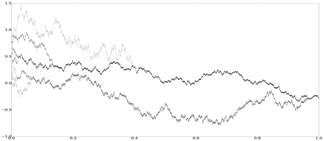


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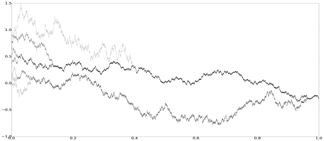

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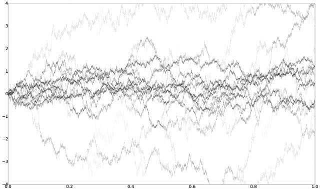

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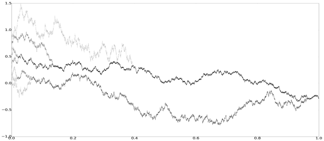

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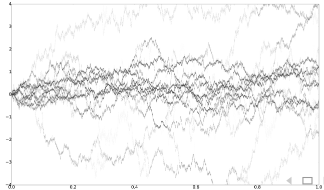

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Thank you!