Degree corrected stochastic block model: excursion representation

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Abstract

This is the first of two complementary works in which we analyze the connected components of the degree-corrected stochastic block model (DCSBM). Our model is a random graph with an underlying community structure and degree in-homogeneity. It belongs to a class of non-rank one models. The scaling limit of connected component sizes in the near-critical regime, obtained by Konarovskyi and Limic (2021) for a subfamily of DCSBM, is non-trivially different (although related to) the standard eternal multiplicative coalescent of Aldous (1997).

The Aldous (1997) excursion representation combined with weak convergence approach to the scaling limits of connected components of random graphs proved to be much more difficult (and therefore rare) for non rank-one models. In this work we show how to build a random field encoding for the connected component structure of DCSBM, in part relying on the theory of Chaumont and Marolleau (2020). We then show how one can, under additional assumptions, reformulate the minimization problem stated in terms of multidimensional first hitting times into an equivalent minimization problem stated for a single real-valued stochastic process. This reformulation relies on a novel composition-like operator on pairs of compatible non-decreasing rcll functions, which might be of independent interest.

Keywords. degree-corrected stochastic block model, breadth-first walk, multidimensional stopping time, excursion representation, composition-like operator

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1 Introduction

Over the past several decades, random graphs have become an indispensable tool for studying real-world networks [38, Chapter 1]. Real-world networks are frequently both large and complicated so that a precise description is near impossible in practice. In an attempt to understand finer properties of these large networks, one typically constructs a family of finite random graph models, and studies various structural properties of these random graphs as n gets large.

A fundamental question in this area is to understand the conditions under which the above large random graphs contain a connected component of size comparable to the size of the entire network. Ever since the fundamental work of Erdős and Rényi, one approaches this problem as follows. Let us denote by G_n the nth element of our sequence of random graphs.

The edge density of G_n is parametrized by $\theta \geq 0$ (for example, as in bond percolation), so that our growing family of random graph families is $((G_n^\theta, \theta \in [0,1]); n \geq 1)$. The aim is to identify the "giant component" phase transition, or more precisely, to find θ_c such that, if $\theta > \theta_c$ then the largest connected component of G_n^θ is of size $\Theta(n)$ with overwhelming probability as $n \to \infty$, and otherwise if $\theta < \theta_c$ then largest connected component of G_n^θ is of size $o(n)$ with overwhelming probability as $n \to \infty$. Understanding the structure of large networks in the near-critical regime, is then naturally related to the problem of understanding the connected components in the parameter window $\theta = \theta_c \pm \varepsilon_n$, for some vanishing model-dependent sequence $(\varepsilon_n)_n$. Bollabás, Janson and Riordan in [12] identify the critical threshold θ_c under rather general hypotheses on the random graph model, however they do not provide any insight into the connected component structure within the (near)-critical window.

For the near-critical Erdős-Rényi random graph, the latter analysis was carried out already by Aldous in [4]. Aldous' approach was based on the breadth-first walk encoding of the connected components sizes, which can be summarized as follows: 1) the excursions of the breadth-first walk above the past infimum encode useful information about the connected components of the random graph, and 2) it is meaningful to take the limit as $n \to \infty$ in this coupling, which yields the scaling limit of the connected component sizes. This approach, together with the analysis of Aldous and Limic [5], has proven valuable for understanding the critical window connected component structure for a number of related random graph models. A far from complete list of papers using these ideas is [3, 10, 13, 20, 23, 24, 30, 33, 34, 36]. Until now, this "encoding via a stochastic process" approach has proved successful for *rank-one* models of random graphs. For this class of models the expected adjacency matrix is approximately a rank-one matrix, or equivalently, $\mathbb{P}(i \sim j) \approx \psi(i)\psi(j)$ for any pair of vertices (i,j), where ψ is some model-dependent positive function.

It is not surprising that many complex (real-world) networks are believed not to be of rank-one. Indeed, a fundamental problem in statistics and computer science is to meaningfully separate data into clusters which share certain important characteristics [28]. When the data is a graph, this clustering involves partitioning of the vertex set V into blocks $C_1, C_2, \cdots, C_m$ for some $m \geq 2$, in such a way that the edge density within blocks is high, and the edge density of links transcending blocks is low (or alternatively, the edge density within blocks is low, while the density of edges transcending blocks is high). A well-known random graph model which exhibits non-trivial community structure (or equivalently, the expected adjacency matrix of higher-rank) is the stochastic block model (SBM). The SBM with m blocks is a graph on mn vertices where for each $i \in [m] := \{1, 2, \cdots, m\}$ there are n vertices of type i, and where an edge connects vertex v of type i and vertex u of type j with probability $p_{i,j} = p_{j,i}$, independently over different pairs of vertices. This graph has become an important model for rigorous analysis of network clustering algorithms. We refer an interested reader to the survey of Abbe [1] for more information and precise statements on these theoretical results.

The second and the third author recently identified in [32] a new critical window for the stochastic block model, and carried out the scaling limit analysis akin to that of [5]. The scaling limit of [32] is the so called *interacting multiplicative coalescent*. The techniques used therein do not include an explicit encoding of SBM via a random walk.

The main goal of this paper is to provide an excursion representation for SBM. Our encoding is quite general as it extends, under certain additional assumptions, to the so-called *degree-corrected stochastic block model* [31], which incorporates degree inhomogeneity among the vertices of the same block. Our study relies on a novel (composition alike) operator on (pairs of) real-valued functions on $[0, \infty)$. This construction is natural but somewhat technical, and it is crucial for the scaling limit analysis. The scaling limit for the sizes of connected components

of the degree-corrected stochastic block models will be exhibited in a forthcoming work [19].

2 Model and Results

2.1 Graphical Models

Denote by

$$\ell^2_\downarrow = \left\{ \mathbf{x} = (x_1, x_2, \cdots) : x_1 \geq x_2 \geq \cdots \geq 0, \quad \sum_{j=1}^{\infty} x_j^2 < \infty \right\}.$$

If $\mathbf{w} = (w_1, w_2, \cdots, w_N, 0, 0, \cdots)$ we say that $\mathbf{w}$ is has finite length, and also that N is the length of $\mathbf{w}$, which we write as $\mathrm{len}(\mathbf{w}) = N$. Consider some weight vector $\mathbf{w} = (w_1, w_2, \cdots) \in \ell^2_\downarrow$ of finite length. Given any square summable vector $\mathbf{x}$ with non-negative entries, we write $\mathrm{ord}(\mathbf{x})$ for the decreasing re-ordering of the entries of $\mathbf{x}$.

We recall the inhomogeneous multiplicative random graph of [4, 5]. The graph $\mathcal{G}(\mathbf{w}, q)$ is a graph on $\mathrm{len}(\mathbf{w})$ vertices labeled by $l \in [\mathrm{len}(\mathbf{w})]$ where

$$\mathbb{P}(l \sim r \text{ in } \mathcal{G}(\mathbf{w}, q)) = 1 - \exp(-q w_l w_r).$$

We interpret the value w_l as the *propensity* of the vertex l to form edges. It is often called *the weight* (or *mass*) of vertex l. A natural coupling of $(\mathcal{G}(\mathbf{w}, q))_{q \geq 0}$ can be realized in a usual way (typical for all percolation processes): let the edge between l and r appear according to a Poisson process with parameter/rate $w_l w_r$, independently over all $l \neq r$. Note that in order to keep track of the connected component structure only at a fixed time q, one can equivalently construct the graph $\mathcal{G}(\mathbf{w}, q)$ by attaching a Poisson (with mean $q w_l w_r$) number of edges between vertex l, r. The original (continuous-time) graph is then obtained from this multi-graph by removing any duplicate edges. The Erdős-Rényi (binomial) random graph $G(n, p)$ is the special case, where $\mathbf{w} = (1, 1, \cdots, 1, 0, \cdots)$ with $\mathrm{len}(\mathbf{w}) = n$ and $q = -\log(1 - p)$.

The degree-corrected stochastic block model (DCSBM) can be constructed in a similar fashion, see [31]. Here we fix m finite length vectors $\mathbf{w}^1, \cdots, \mathbf{w}^m \in \ell^2_\downarrow$, and a symmetric $m \times m$ matrix Q with non-negative real entries. Each vertex v is of the form $v = (l, i)$, where $i \in [m]$ is its *type* (this means that v is an element of the ith block) and w^i_l is its assigned weight corresponding to the *propensity* of v to form edges. Let us denote by $\mathbf{W}$ the vector $(\mathbf{w}^1, \cdots, \mathbf{w}^m) \in (\ell^2_\downarrow)^m = \ell^2_\downarrow \times \ell^2_\downarrow \times \cdots \times \ell^2_\downarrow$ listing all the propensities of all the vertices in a type-wise increasing (and propensity-wise non-increasing) ordering. The random (multi-)graph $\mathcal{G}(\mathbf{W}, Q)$ is obtained after attaching $\mathrm{Poi}(Q_{i,j} w^i_l w^j_r)$ many edges between each pair of vertices (l, i) and (r, j), independently over different pairs. Since we are concerned here with the sizes of connected components, all duplicate edges and all self-loops will be ignored.

The DCSBM has two kinds of parameters. Parameters of the first kind are the weight vectors $\mathbf{w}^i$, which give rise to the degree inhomogeneity in the graph. The larger the value of w^i_l, the more neighbors will the corresponding vertex (l, i) have on the average. This is analogous to the setting of the rank-one graph $\mathcal{G}(\mathbf{w}, q)$. Parameters of the second kind are the entries of Q, and they determine the block structure of the graph. The larger the value of $Q_{i,j}$, the more likely will an edge appear between a vertex of type i and a vertex of type j. In fact, the matrix Q is a multi-dimensional analogue of time q in the rank-one model.

Remark 2.1. One could incorporate the information on the diagonal of Q within the weight data. More precisely, define $\mathbf{W}^Q = (\sqrt{Q_{1,1}}\mathbf{w}^1, \cdots, \sqrt{Q_{m,m}}\mathbf{w}^m)$ and $Q'_{i,j} := Q_{i,j}/\sqrt{Q_{i,i}Q_{j,j}}$.

In this way Q' has 1 on the diagonal, and moreover it is easy to see that the open edges (i.e. connections) in $\mathcal{G}(\mathbf{W}, Q)$ and in $\mathcal{G}(\mathbf{W}^Q, Q')$ have the same law. Indeed, in the latter model, the scaling of the block weights cancels out the scaling of the m-dimensional time. Note, however, that in this coupling (visually) the same connected components of $\mathcal{G}(\mathbf{W}, Q)$ and $\mathcal{G}(\mathbf{W}^Q, Q')$ have completely different weights.

2.2 First hitting times of fields

An encoding of the connected components weights of the naturally coupled family of random graphs $(\mathcal{G}(\mathbf{w}, q); q > 0)$ is due to Limic [33]. This construction relies on $\mathrm{len}(\mathbf{w})$ independent exponential random variables $(\xi_l)_{l\in[\mathrm{len}(\mathbf{w})]}$, where $\xi_l \sim \mathrm{Exp}(w_l)$. Here and below $\mathrm{Exp}(c)$ denotes an exponential random variable with rate c (mean $\frac{1}{c}$). If A is a collection of vertices in $\mathcal{G}(\mathbf{w}, q)$, define the *weight of* A to be $\sum_{l\in A} w_l$. For each q, denote by $\mathscr{M}_q(1) \geq \mathscr{M}_q(2) \geq \cdots$ the weights of the connected components of $\mathcal{G}(\mathbf{w}, q)$, listed in non-increasing order.

For $q > 0$, let $X^q \equiv X^{q,\mathbf{w}} = (X^q(t); t \geq 0)$ denote the random walk-like process

$$X^q(t) = -t + \sum_{l=1}^{\mathrm{len}(\mathbf{w})} w_l 1_{[\frac{1}{q}\xi_l \leq t]}. \tag{2.1}$$

For $y \geq 0$, let $T^q(y) = \inf\{t : X^q(t-) = -y\} \equiv \inf\{t : X^q(t-) \leq -y\}$ and denote by $Y_1^q < Y_2^q < \cdots$ the successive (finitely many) jump times of T^q viewed as a process in y. A key result of [33] is its Proposition 5, which states (in a slightly different language) that the processes $(\mathrm{ord}(T^q(Y_l+) - T^q(Y_l); l \geq 1))_{q>0}$ and $(\mathscr{M}_q(1), \mathscr{M}_q(2), \cdots)_{q>0}$ are identical in law. The main advantage of this encoding over similarly looking ones in [4, 5], and various analogues constructed in the meantime, is that it works on the level of processes. A different full encoding for connected component sizes of random graphs with (or without) deletion was invented by Martin and Ráth in [34].

It was observed already in [4, 5] that, for each fixed q, the walk-based ordered encoding $(T^q(Y_l+) - T^q(Y_l); l \geq 1)$ is distributed as a size-biased copy of $\mathrm{ord}(T^q(Y_l+) - T^q(Y_l); l \geq 1)$. Let $(\mathscr{M}^*(l); l \geq 1)$ be a size-biased reordering of $(\mathscr{M}(l); l \geq 1)$, where the size of $\mathscr{M}(l)$ is equal to its weight.

Corollary 2.2 (see also [33, Proposition 1])**.** *For each* $q > 0$

$$(T^q(Y_l+) - T^q(Y_l); l \geq 1) \stackrel{d}{=} (\mathscr{M}^*(l); l \geq 1)\,.$$

Proof. Since $\mathrm{ord}(T^q(Y_l+) - T^q(Y_l); l \geq 1)$ and $(\mathscr{M}_q(1), \mathscr{M}_q(2), \cdots)$ are equally distributed, the same is true for their respective size (weight)-biased lists. □

We now present a generalization of this representation, based on several ideas in the random tree and branching process literature [6, 15, 16]. The set-up is as follows: recall $\mathbf{W}, Q$ fixed above, and provided that $w_l^i > 0$, we let ξ_l^i have $\mathrm{Exp}(w_l^i)$ distribution, where all the variables in the family $(\xi_l^i)_{i\in[m], l\geq 1, w_l^i>0}$ are independent. When we refer to a vertex $v = (l, i)$ we will often simply write ξ_v in place of ξ_l^i. For all $i, j \in [m]$, let us define

$$R_{i,j} := Q_{i,j}/Q_{i,i}, \tag{2.2}$$

and so that in particular $R_{i,i} \equiv 1$. In addition, for each $i, j \in [m]$ and all $t \geq 0$ we define

$$X_{i,j}(t) = \begin{cases} -t + \sum_{l=1}^{\text{len}(\mathbf{w}^j)} w_l^j 1_{[\frac{1}{Q_{j,j}}\xi_l^j \leq t]}, & \text{if } i = j, \\ R_{i,j} \sum_{l=1}^{\text{len}(\mathbf{w}^j)} w_l^j 1_{[\frac{1}{Q_{j,j}}\xi_l^j \leq t]}, & \text{if } i \neq j. \end{cases} \tag{2.3}$$

The processes $X_j = (X_{1,j}, \cdots, X_{m,j})$, $j \in [m]$, clearly depend on both Q and $\mathbf{W}$; however, in the sequel this fact will be mostly suppressed from the notation. Observe that the vector-valued processes X_j are independent over j. Also observe that for each fixed j, the off-diagonal processes depend deterministically on the diagonal $(X_{j,j}(t);\, t \geq 0)$. In particular, for any given $j \in [m]$, all the processes $(X_{i,j};\, i \in [m])$ have simultaneous jumps.

In order to state and prove an analogue of Corollary 2.2, we need to define an analogue of the first hitting times process $(T^q(y);\, y \geq 0)$. Such processes were studied recently by Chaumont and Marolleau in [16, 17] in the context of random fields. We now recall the setting of [16, 17], as well as some of their results which are fundamental for the present study. Given (deterministic) càdlàg functions $x_{i,j}$ for $i, j \in [m]$ such that $x_{i,j}(0) = 0$ for all i, j, and such that $x_{i,j}$ is non-decreasing when $i \neq j$ and $x_{i,i}(t) - x_{i,i}(t-) \geq 0$ for all t and $i \in [m]$, let us consider the following field

$$\mathbb{x}(\vec{t}) = \mathbb{x}(t_1, \cdots, t_m) = \left(\sum_{j=1}^m x_{1,j}(t_j), \cdots, \sum_{j=1}^m x_{m,j}(t_j) \right). \tag{2.4}$$

It is proved in [16] that for each $\vec{y} \in \mathbb{R}_+^m$ there exists a unique solution to

$$\begin{cases} x_i(\vec{t}-) = \sum_{j=1}^m x_{i,j}(t_j-) = -y_i, \quad \forall i \text{ such that } t_i < \infty, \\ \vec{t} \to \min. \end{cases} \tag{2.5}$$

Let us denote by

$$\mathbb{T}(\mathbb{x}; \vec{y}) = (T_1(\mathbb{x}; \vec{y}), T_2(\mathbb{x}; \vec{y}), \cdots, T_m(\mathbb{x}; \vec{y})) \in [0, \infty]^m \tag{2.6}$$

this unique minimal solution. The condition $\vec{t} \to \min$ means that any other solution $\vec{t'}$ to (2.5) must be component-wise greater or equal to $\mathbb{T}(\mathbb{x}; \vec{y})$, or equivalently that $T_j(\mathbb{x}; \vec{y}) \leq t'_j$ for all $j \in [m]$ (which we also write as $\mathbb{T}(\mathbb{x}; \vec{y}) \leq \vec{t'}$).

Remark 2.3. Observe that $\vec{t} = (\infty, \infty, \cdots, \infty)$ is always a solution to the equation in (2.5). As we will soon see, most of the random fields relevant for our present study will be such that $\mathbb{T}(\mathbb{x}; \vec{y})$ takes finite values in $\mathbb{R}_+^m$, for all $\vec{y} \in \mathbb{R}_+^m$ almost surely.

In analogy to the deterministic setting, we now consider the $\mathbb{R}^m$-valued and $\mathbb{R}_+^m$-indexed field $\mathbb{X} = \mathbb{X}^{\mathbf{W},Q} = (\mathbb{X}(\vec{t}); \vec{t} \in \mathbb{R}_+^m)$, defined by

$$\mathbb{X}(\vec{t}) = \left(\sum_{j=1}^m X_{1,j}(t_j), \cdots, \sum_{j=1}^m X_{m,j}(t_j) \right). \tag{2.7}$$

By abuse of language we will henceforth refer to vectors $\vec{t} \in \mathbb{R}_+^m$ as "time", or less-frequently as "time-lines".

2.3 Random field encoding of DCSBM

Recall that $\mathbf{W}$ and Q are fixed as above. Our next goal is to encode the weights of the connected components of the graph $\mathcal{G} = \mathcal{G}(\mathbf{W}, Q)$ in terms of its corresponding $\mathbb{X}$. Let us list the connected components $\mathcal{C}(1), \mathcal{C}(2), \cdots$ of $\mathcal{G}$ in some arbitrary (measurable) fixed way.

Recall that each vertex v in $\mathcal{G}$ is identified with (l, i) for some $l \geq 1$ and $i \in [m]$, where i is the type of v, and l is the ranking of v's propensity or weight (specified as w_l^i) among all the type i vertices in $\mathcal{G}$. In forthcoming calculations it will often be convenient to write $\mathbf{t}(v)$ to mean i, the type of v. We can therefore define the total weight of type j vertices in the rth connected component $\mathcal{C}(r)$ of $\mathcal{G}$ by

$$\mathscr{M}_j(r) := \sum_{l:(l,j)\in\mathcal{C}(r)} w_l^j.$$

To keep track of this information we use a family of m-dimensional random vectors

$$\vec{\mathscr{M}}(r) = (\mathscr{M}_1(r), \cdots, \mathscr{M}_m(r)), \quad r \geq 1. \tag{2.8}$$

We will encode the family $(\vec{\mathscr{M}}(r); r \geq 1)$ via the family of the first hitting times $(\mathbb{T}(\mathbb{x}; \vec{y}); \vec{y} \in L)$ along a line $L \subset \mathbb{R}_+^m$. More precisely, let us fix a vector $\vec{\rho} \in \mathbb{R}_+^m \setminus \{\vec{0}\}$ and consider the half-line $L = \{\vec{\rho}y; y \geq 0\}$ in the direction of $\vec{\rho}$. Define the vector-valued process

$$\mathbf{T} = \mathbf{T}^{\vec{\rho},\mathbf{W},Q} = (\mathbf{T}(y); y \geq 0),$$

by letting

$$\begin{aligned} \mathbf{T}(y) = \mathbb{T}(\mathbb{X}; \vec{\rho}y) &= \inf\{\vec{t} : X_i(\vec{t}-) = -\rho_i y, \ \ \forall i \text{ s.t. } t_i < \infty\} \\ &= \inf\{\vec{t} : \mathbb{X}(\vec{t}-) = -\vec{\rho}y\}. \end{aligned} \tag{2.9}$$

Remark 2.4. For a fixed y, $\mathbf{T}(y)$ is analogous to the above deterministic minimizer $\mathbb{T}(x; \vec{\rho}y)$, except that here we are (almost) sure that there is a finite random quantity $\vec{S}^y$ such that $\sum_j X_{i,j}(\vec{S}_j^y-) = -\rho_i y$ for each $i \in [m]$, or equivalently, that $T_i(\mathbb{X}; \vec{\rho}y) < \infty$ almost surely for each $i \in [m]$ and all $y \geq 0$. This is due to the fact that $X_{i,j}$ remains bounded for all $i \neq j$, while $X_{i,i}(t) \longrightarrow -\infty$ as $t \to \infty$.

It is easy to see that, with probability one, the process $y \mapsto \mathbf{T}(y)$ is non-decreasing with left-continuous paths. By the construction of the minimal solution $\mathbb{T}$ in the proof of [16] Lemma 2.3, one can see that $\mathbf{T}(y)$ is a (multi-dimensional) stopping time with respect to the filtration generated by $\mathbb{X}$. Furthermore, there are at most $\sum_{j=1}^m \text{len}(\mathbf{w}^j)$ many jumps of the process $\mathbf{T}$. This is due to the construction (2.3)–(2.7) (in particular, there are $\text{len}(\mathbf{w}^j)$ many jumps of $X_{j,j}$, for each $j \in [m]$), joint with the fact that (in our discrete setting, analogously to the $m = 1$ setting) to each jump time Y of $\mathbf{T}$ corresponds a random index $J \in [m]$, and a unique jump of $X_{J,J}$ (on the Jth timeline, say at time $S_J(Y)$) such that $X_{J,J}$ starts an excursion at $S_J(Y)$. See also Remark 2.7.

Definition 2.5. Let $(\vec{\Delta}(r); r \geq 1)$ denote the jump sizes of $y \mapsto \mathbf{T}(y)$ listed in chronological order.

Recall the matrix R defined in (2.2), and recall that $(\mathcal{C}(r); r \geq 1)$ is an arbitrary ordering of the connected components of $\mathcal{G}$. Recall that $\mathbf{t}(v)$ is the type of vertex v. Given a set of vertices $\mathcal{A}$ of $\mathcal{G}$, let us assign to $\mathcal{A}$ its $(\vec{\rho}, Q)$-*scaled mass*, or *scaled mass* for short, as

$$\mathscr{S}(\mathcal{A}) \equiv \mathscr{S}(\mathcal{A}; \vec{\rho}, Q) := \sum_{v\in\mathcal{A}} \rho_{\mathbf{t}(v)} Q_{\mathbf{t}(v),\mathbf{t}(v)} w_v. \tag{2.10}$$

Then, for each r, the scaled mass of $\mathcal{C}(r)$ simplifies to

$$\mathscr{S}(\mathcal{C}(r)) \equiv \mathscr{S}(\mathcal{C}(r); \vec{\rho}, Q) := \sum_{i=1}^{m} \rho_i Q_{i,i} \mathscr{M}_i(r), \tag{2.11}$$

where $\vec{\mathscr{M}}(r)$ is from (2.8). It may be more accurate to write $\mathscr{S}(V(\mathcal{C}(r)))$ as this quantity depends on the *vertex set* $V(\mathcal{C}(r))$ for the component $\mathcal{C}(r)$, but we think this is too cumbersome of notation. Let $(\mathcal{C}^*(r); r \geq 1)$ denote a size-biased reordering of $\{\mathcal{C}(r) : \mathscr{S}(\mathcal{C}(r)) > 0\}$ by their scaled mass. For each r, we denote by $\vec{\mathscr{M}}^*(r)$ the corresponding weight vector of the component $\mathcal{C}^*(r)$. The following is our first main result.

Theorem 2.6. *Let* $\mathbf{W}$, Q *be fixed as above. Then, for each* $\vec{\rho} \in \mathbb{R}_+^m \setminus \{\vec{0}\}$, *we have the identity*

$$\left(\vec{\Delta}(r);\, r \geq 1\right) \stackrel{d}{=} \left(R\vec{\mathscr{M}}^*(r);\, r \geq 1\right). \tag{2.12}$$

Its proof is postponed until Section 4.5.

Let us define

$$[m]_\rho := \{k \in [m] : \rho_k > 0\}. \tag{2.13}$$

Given a set $\mathcal{A}$ of vertices, and a set $I \subset [m]$ of indices we abuse notation and write

$$\mathcal{A} \cap I \equiv \mathcal{A} \cap (\mathbb{N} \times I) = \{v \in \mathcal{A} : \mathbf{t}(v) \in I\}. \tag{2.14}$$

Remark 2.7. On the event $\{\mathcal{C}(l) \cap [m]_\rho = \emptyset\}$, neither $\mathcal{C}(l)$ nor its corresponding vector $\vec{\mathscr{M}}(l)$ appear in the size-biased list above. The encoding via field $\mathbb{X}$ cannot access any such $\mathcal{C}(l)$ (since the exploration is done only in the direction of $\vec{\rho}$), and therefore the scaled mass of $\mathcal{C}(l)$ will not appear in the list on the RHS of (2.12). Concerning the list on the LHS of (2.12), we wish to point out that. as the proof of Theorem 2.6 will show, the random process $\mathbf{T}(y)$ can be written as

$$\vec{\rho} y + \sum_{x<y} (\mathbf{T}(x+) - \mathbf{T}(x)) = \vec{\rho} y + \sum_{r \in J_y} \vec{\Delta}(r), \qquad y \geq 0,$$

for some finite and uniformly bounded set of jumps J_y. Therefore, $T_j(+\infty) := \lim_{y\to\infty} T_j(y)$ and it is almost surely finite, if and only if, $\rho_j = 0$. If $\rho_j = 0$, then the time $T_j(+\infty)$ may (and typically does) appear *before* some of the excursions (above past infimum) of the process $X_{j,j}$ even begin. The information contained in $X_{j,j}$ on $[T_j(+\infty), +\infty)$, for all $j \in [m] \setminus [m]_{\vec{\rho}}$, could probably be used to reconstruct the connected component sizes of the DCSBM intersected with $[m] \setminus [m]_\rho$, however it is not clear if this extra effort would bring any significant benefits. We will exhibit an encoding of the connected component sizes in each probe direction $\vec{\rho}$. By varying $\vec{\rho}$, one can access all the connected components of the DCSBM.

2.4 From fields to processes

Theorem 2.6 is a random field generalization of Corollary 2.2. Let us first consider a restatement of Corollary 2.2 in terms of the excursions of the process X^q.

Recall that if $f : [0, \infty) \to \mathbb{R}$ is a càdlàg function, an interval (l, r) is called an *excursion (above past infima)* interval if

$$\inf_{t \leq l} f(t) = \inf_{t \leq r} f(t) \qquad \text{and} \qquad f(s-) > \inf_{t \leq r} f(t) \text{ for all } s \in (l, r).$$

For a function f, we will denote by $\mathcal{E}(f)$ the collection of excursions (above past infima), and by $\mathcal{L}(f)$ the multiset of excursion lengths $\{r - l : (l, r) \in \mathcal{E}(f)\}$. Finally, we let

$$\mathcal{L}^{\downarrow}(f) \text{ be the non-increasing rearrangement of } \mathcal{L}(f),$$

provided that it is well-defined (if and only if there are at most finitely many excursions of f longer than δ, for each $\delta > 0$).

As discussed in the introduction, the pioneering work of Aldous [4] was a base to a number of studies. With a representation analogous to Corollary 2.2 and a corresponding scaling limit for the sequence of appropriately rescaled process $(X^q)_q$ one can often appeal to a quite general theory [4, Lemma 7 and Proposition 15] to conclude relatively easily that the rescaled component weights (of the random graph under consideration) converge in distribution to a random element of $\ell^2_{\downarrow}$ as the size of the graph diverges. This general approach by Aldous [4] is based on relating the excursions of the prelimiting processes with excursions of the limiting stochastic process. While the construction of the first hitting times $\mathbf{T}(y)$ in [16] is a very useful tool for our analysis, it does not give much insight into the behaviour of the field "between" $\mathbf{T}(y)$ and $\mathbf{T}(y+)$. So it is not clear what a reasonable definition of an excursion would even be in the present context.

A major contribution of this paper is a construction of a single curve $\vec{\gamma} : \mathbb{R}_+ \to \mathbb{R}^m_+$ which *combines the information on the m time-lines in an appropriate way* so that the first hitting times of the $\mathbb{R}^m_+$-indexed and $\mathbb{R}^m$-valued field match the first hitting times of a $\mathbb{R}_+$-indexed real-valued process (as it turns out, there are several such processes). A precise statement is the following theorem. (We prove this result under weaker assumptions, which are cumbersome to state at this point.)

Theorem 2.8. *Suppose that $\mathbb{x}$ is a field as in (2.4), $\inf_{s\le t} x_{i,i}(s) < 0$ for all $t > 0$ and $\liminf_t x_{i,i}(t) = -\infty$ and that there exists some vector $\vec{\rho} = (\rho_1, \dots, \rho_m) \in (0,\infty)^m$ such that for each $l \in [m]$ and all $i, j \neq l$*

$$\frac{x_{i,l}(t)}{\rho_i} = \frac{x_{j,l}(t)}{\rho_j} \text{ for all } t \ge 0. \tag{2.15}$$

Let $\mathbf{T}(y) = \mathbb{T}(\mathbb{x}; \vec{\rho}y)$ be the first hitting time of level $-\vec{\rho}y$ for $\mathbb{x}$. Then, there exists a Lipschitz curve $\vec{\gamma} : \mathbb{R}_+ \to \mathbb{R}^m_+$ with non-decreasing coordinates such that

(1) $\|\vec{\gamma}(s)\|_1 = s$ for every $s \ge 0$.

Moreover, for all $y \ge 0$, if $\mathbf{T}(y) \in \mathbb{R}^m_+$ then

(2) $\vec{\gamma}(\|\mathbf{T}(y)\|_1) = \mathbf{T}(y)$ and

(3) for all $i \in [m]$

$$\inf\left\{ s \ge 0 : \sum_{j=1}^m x_{i,j} \circ \gamma_j(s-) = -\rho_i y \right\} = \|\mathbf{T}(y)\|_1.$$

An immediate consequence of Theorems 2.6 and 2.8 is the following result.

Theorem 2.9. *Let $\mathbf{W}$, Q, R and $\mathbb{X}$ be as in Theorem 2.6, and let $\mathcal{G}(\mathbf{W}, Q)$ be the corresponding DCSBM. Suppose that, in addition,*

$$R_{i,j} = \frac{Q_{i,j}}{Q_{j,j}} = \rho_i \nu_j, \text{ for all } i \neq j, \tag{2.16}$$

for some vectors $\vec{\rho}, \vec{\nu} \in (0,\infty)^m$. Recall that $(\vec{\mathscr{M}}(r); r \ge 1)$ are the vector-valued component weights of $\mathcal{G}(\mathbf{W}, Q)$ arranged in some arbitrary order. Then there exists a vector valued curve $\vec{\gamma}$ with non-decreasing coordinates such that

1. *The ordered excursion lengths of* $\sum_{j=1}^m X_{i,j} \circ \gamma_j(t)$ *are equal in law to the reording of* $(\|R\vec{\mathscr{M}}(r)\|_1; r \geq 1)$*; i.e.*

$$\mathcal{L}^{\downarrow}\left(\sum_{j=1}^m X_{i,j} \circ \gamma_j\right) \stackrel{d}{=} \mathrm{ORD}\left(\|R\vec{\mathscr{M}}(r)\|_1; r \geq 1\right).$$

2. *If* $((l_p, r_p); p \geq 1)$ *are the excursion intervals of* $\sum_{j=1}^m X_{i,j} \circ \gamma_j$ *arranged chronologically, then*

$$(\vec{\gamma}(r_p) - \vec{\gamma}(l_p); p \geq 1) \stackrel{d}{=} \left(R\vec{\mathscr{M}}^*(p); p \geq 1\right).$$

2.5 Comments on models with condition (2.16)

2.5.1 Restrictions with few blocks

Let us begin by noting that whenever there are two blocks (i.e. $m = 2$) the condition (2.16) is always true provided that $Q_{i,j} > 0$ for all $i, j \in [2]$. In fact, (2.15) in Theorem 2.8 is always true in the case where there are just two types.

Assumption (2.16) starts to become more interesting in the case where $m = 3$. By first examining (2.15) in Theorem 2.8 as well as the form of the field $\mathbb{X}$ in (2.3), we see that for any Q we can set

$$\rho_1 = \frac{Q_{1,2}Q_{1,3}}{Q_{1,1}} \qquad \rho_2 = \frac{Q_{2,1}Q_{2,3}}{Q_{2,2}} \qquad \rho_3 = \frac{Q_{3,1}Q_{3,2}}{Q_{3,3}}$$

and

$$\nu_1 = \frac{1}{Q_{2,3}} \qquad \nu_2 = \frac{1}{Q_{1,3}} \qquad \nu_3 = \frac{1}{Q_{1,2}}.$$

Indeed, looking at distinct $i, j, k \in [3]$ we have

$$\rho_i \nu_j = \frac{Q_{i,j}Q_{i,k}}{Q_{i,i}} \cdot \frac{1}{Q_{i,k}} = \frac{Q_{i,j}}{Q_{i,i}} = R_{i,j}.$$

In particular, provided that Q is a symmetric matrix with strictly positive entries Theorem 2.9 is always applicable for particular (and explicit) choices of $\vec{\rho}$ and $\vec{\nu}$.

A simple dimension counting argument implies that (2.16) can not be satisfied in great generality for $m > 3$. Indeed, the collection of symmetric $m \times m$ matrices Q with positive entries forms a $\binom{m+1}{2}$ dimensional manifold, while the collection of matrices Q that satisfy (2.16) is only of dimension $3m$ (m for the diagonal entries of Q and m for each the vectors $\vec{\rho}, \vec{\nu}$).

2.5.2 Link with [32]

Condition (2.15) is equivalent to (2.16) in our stochastic setting, and we furthermore have an interesting probabilistic interpretation.

Lemma 2.10. *The symmetric matrix* Q *and the vector* $\vec{\rho} \in (0, \infty)^m$ *satisfy* (2.16) *if and only if there exist* $q_i > 0$, $i \in [m] \cup \{0\}$, *such that vertices* (l, i) *and* (k, j) *of* $\mathcal{G}(\mathbf{W}, Q)$ *are connected by an edge with probability* $1 - e^{-q_0 \rho_i w_l^i \rho_j w_k^j}$ *if* $i \neq j$ *and with probability* $1 - e^{-q_i \rho_i w_l^i \rho_j w_k^j}$ *if* $i = j$.

Proof. Since Q is a symmetric matrix, due to (2.16) we have for all $i \neq j$

$$Q_{j,j}\rho_i\nu_j = Q_{i,i}\rho_j\nu_i \iff \frac{Q_{j,j}\nu_j}{\rho_j} = \frac{Q_{i,i}\nu_i}{\rho_i}.$$

Let $q_0 := Q_{1,1}\nu_1/\rho_1 = Q_{j,j}\nu_j/\rho_j$, $j \in [m]$, and $q_i := Q_{i,i}/\rho_i^2$, $j \in [m]$. It is straight-forward to see that $Q_{i,j} = q_0\rho_i\rho_j$ for $i \neq j$ and $Q_{i,i} = q_i\rho_i^2$. □

The above lemma gives a connection with the setting of [32]. The class of models satisfying (2.16) includes the SBMs analyzed in [32], but it is much larger since it allows for varying intra-block connection probabilities over types. This connection will be important in our sequel paper [19].

2.5.3 An epidemiological interpretation

Let us now describe possible epidemiological interpretation. We have a population of $N = \sum_{j=1}^m \mathrm{len}(\mathbf{w}^j)$ many individuals segmented into m many sub-types. Each individual (l, j) of type j has some propensity w_l^j of both catching or transmitting a disease to their neighbors. The factor ν_j represents the propensity of a type j individual to transmit the disease to others, for example by not taking preventative measures to stop the spread of the disease. Finally, there is some likelihood that type j individuals come into contact with type i individuals which is represented by

$$\frac{\tilde{\rho}_i\tilde{\rho}_j}{\tilde{\rho}_i + \tilde{\rho}_j} \qquad \text{where} \qquad \tilde{\rho}_i = Q_{i,i}\rho_i.$$

To model a disease spreading through the population we can use a direct graph where a direct edge from u to v means individual u infected individual v. Moreover our graphs is built by independently adding an edge from (l, i) to (r, j) with probability

$$1 - e^{-w_l^i w_r^j \frac{\tilde{\rho}_i\tilde{\rho}_j}{\tilde{\rho}_i+\tilde{\rho}_j}\nu_j}, \qquad \forall i \neq j.$$

Forgetting direction of the edges, we see that an (undirected) edge between (i, l) and (j, r) appears with the probability $1 - e^{-p}$, where

$$\begin{aligned} p &= w_l^i w_r^j \left(\frac{\tilde{\rho}_i\tilde{\rho}_j}{\tilde{\rho}_i + \tilde{\rho}_j}\nu_j + \frac{\tilde{\rho}_j\tilde{\rho}_i}{\tilde{\rho}_i + \tilde{\rho}_j}\nu_i \right) \\ &= w_l^i w_r^j \left(\frac{Q_{ij}\tilde{\rho}_j}{\tilde{\rho}_i + \tilde{\rho}_j} + \frac{Q_{ji}\tilde{\rho}_i}{\tilde{\rho}_i + \tilde{\rho}_j} \right) = w_l^i w_r^j Q_{ij} \qquad \forall i \neq j. \end{aligned}$$

Thus the (weakly) connected components in this disease model are equal in law to the connected components that Theorem 2.9 can analyze.

3 Discussion

3.1 Past and related work

To the best of our knowledge, the "stochastic process encoding" for analyzing the connected components of critical random graphs which are not rank-1 appeared until now only a few times in the literature. The first such work is by Dembo, Levit and Vadlamani [22] on the so-called quantum Erdős-Rényi (QER) random graph. In this model, each vertex in the standard

Erdős-Rényi graph is replaced by a copy of a circle S^1 cut into arcs according to a Poisson process, and these arcs then become the vertices of the QER random graph. Edges are included subsequently according to another independent Poisson process.

A more closely related model to ours appears in the works of Federico [27] and Wang [39] on the near-critical bipartite Erdős-Rényi random graph. While the actual explorations used in these papers differ from ours, their encodings correspond to a join of two separate explorations (one explores the left-vertex set, and the other the right-vertex set of the bipartite graph) into a single stochastic process, which can be analyzed via weak convergence techniques. As mentioned above in Section 2.5.1, our encoding is completely general in the rank-2 case, whenever $Q_{i,j} > 0$ for all i, j. Therefore our encoding does not technically cover the bipartite case where $Q_{i,i} = 0$ for both i and so we cannot encode the graphs studied by [27, 39]. However, by taking the intra-block connection probabilities sufficiently small and using the result of Janson [29, Corollary 2.12], one can see that the bipartite ER graph is asymptotically equivalent to a model with $Q_{i,i} > 0$ and therefore one we can encode. See also [38, Section 6.7]. This approach is taken by DC in [18] to analyze the general rank-2 multiplicative random graphs.

A different approach has been quite successful for analyzing other classes of non-rank-1 random graphs. A general method for proving that the connected components of certain critical random graphs, viewed *as metric measure spaces*, lie in the *basin of attraction* of the continuum limit of critical Erdős-Rényi random graphs of Addario-Berry et al. [2, 3], was developed by Bhamidi et al. in [9]. Roughly speaking, this method consists in showing that the barely subcritical random graph satisfies certain asymptotic properties (this gives the "blobs" of [9]), and that the evolution of the model from the barely subcritical to the critical regime is approximately that of the Aldous standard multiplicative coalescent [4] (giving the "blob-level superstructure" of [9]) and converge to the continuum random graph [10]. This program has more recently enabled Blanc-Renaudie et al. [11] (resp. Bhamidi et al. [7]) to prove that the connected components of the near-critical percolation on the d-dimensional hypercube (resp. on a graph converging to an L^3-graphon) converge to the continuum random graph of [2, 3]. It is not likely that this approach would apply in our setting, which is more closely related to the restricted multiplicative merging and the interacting eternal multiplicative coalescents of [32], than to the Aldous standard multiplicative coalescent.

In addition to the aforementioned papers, several works used exploration processes and their related height processes (constructed by Duquesne and Le Gall in [25]) for analyzing scaling limits of multi-type Galton-Watson trees. In [35], Miermont introduces a "reduction of types" argument to show that (modulo some scaling) the height process of a critical multitype Galton-Watson forest with finite variance converges to a reflected Brownian motion (which also encodes the limit for a single type Galton-Watson forest). This result was generalized in the case of offspring distributions in the domain of attraction of an α-stable random variable by Berzunza [8], and in the case of infinitely many types by de Raphélis [21].

3.2 Future work

As mentioned in the Introduction, in this report we initiate our study of the degree corrected stochastic block model, which is continued in our work in progress [19]. In this paper we lay out the encoding of the graph model via a random field, and (under additional assumptions) develop a technique for transforming the field encoding into an encoding by a conventional real-valued stochastic process.

Our subsequent work [19] is concerned with scaling limits. Define $\sigma_r(\mathbf{x}) := \sum_{p=1}^{\infty} x_p^r$. More precisely, we study the behaviour of the sequence of graphs $\mathcal{G}(\mathbf{W}^{(n)}, Q^{(n)})$, as $n \to \infty$,

under the following asymptotic conditions on $\mathbf{W} = \mathbf{W}^{(n)}$ and $Q = Q^{(n)}$: there exist sequences $0 < a_n \to 0$, $\vec{\rho}^{(n)} \to \vec{\rho} \in (0,\infty)^m, \vec{\nu}^{(n)} \to \vec{\nu} \in (0,\infty)^m$, and for each $i \in [m]$ there exist $\alpha_i > 0, \beta_i \geq 0, \alpha_i, \beta_i, \lambda_i \in \mathbb{R}$, and $\mathbf{c}_i = (c_{i,1}, c_{i,2}, \cdots) \in \ell^3_\downarrow$, such that for each $i \in [m]$ and $j \neq i$

$$\frac{w_p^{(n),i}}{\sigma_2(\mathbf{w}^{(n),i})} \to c_{i,p}, \qquad \text{and } \sigma_2(\mathbf{w}^{(n),i}) \to 0, \tag{3.1}$$

$$\frac{\sigma_3(\mathbf{w}^{(n),i})}{\sigma_2(\mathbf{w}^{(n),i})^3} \to \beta_i + \sigma_3(\mathbf{c}_i), \qquad \text{where } \beta_i > 0 \text{ or } \sigma_2(\mathbf{c}^i) = +\infty, \tag{3.2}$$

$$\frac{\sigma_2(\mathbf{w}^{(n),i})}{a_n} \to \alpha_i, \qquad \text{and } \frac{R^{(n)}_{i,j}}{a_n} = \frac{Q^{(n)}_{i,j}}{a_n Q^{(n)}_{j,j}} = \frac{\rho_i^{(n)}\nu_j^{(n)}}{a_n}, \tag{3.3}$$

$$\text{and in addition } Q^{(n)}_{i,i} = \frac{1}{\sigma_2(\mathbf{w}^{(n),i})} + \lambda_i + o(1). \tag{3.4}$$

Hypotheses (3.1)-(3.2) and (3.4) are the well-known conditions arising from [5]. Informally, the left-hand side of (3.3) guarantees that the weights of all the type i vertices are roughly of the same order, while the right-hand side is a technical condition which allows us to apply the results obtained in Sections 5 and 6 of the present work. One can also check that these assumptions are the natural inhomogeneous generalizations of [32].

From hypotheses (3.1)–(3.4) (without using the RHS in (3.3)) it is not hard (applying results from [5, 33]) to derive the scaling limit for each of the processes $X^{(n)}_{k,i}$ in (2.3), and moreover the joint scaling limit for the matrix-valued process $(X_{i,j})_{i,j\in[m]}$. In addition, we already obtained an encoding of the finite graph, so one might think that the scaling limit theorem for our DCSBM model would be a standard extension of [4, 5]. However, we face several non-trivial technical obstacles when passing from the random fields $\mathbb{X}^{(n)}$ to the stochastic processes $\sum_{i=1}^m X^{(n)}_{k,i} \circ \gamma^{(n)}_i(t)$, $k \in [m]$. Without going into details, here we give an indication of our approach developed in [19].

- Given a deterministic sequence of fields $\mathbb{x}^{(n)} = (x^{(n)}_{i,j})_{i,j\in[m]}$, $n \geq 1$, converging to the field $\mathbb{x}$, where each $\mathbb{x}^{(n)}$ satisfies the hypotheses of Theorem 2.8, we establish convergence, under appropriate assumptions, for the sequence of curves $\vec{\gamma}^{(n)}$ (constructed via Theorem 2.8).

- Since there is no total order on the space of $\mathbb{R}^m_+$-valued vectors, we construct a Polish space $\widetilde{\ell^{2,m}}$, which is an analog of $\ell^2_\downarrow$ for sequences of vectors with square summable norms. This construction is analogous to the construction of the space of graphons [14].

- We establish tightness in $\widetilde{\ell^{2,m}}$ for the sequence of vector-valued connected component weights of $\mathcal{G}(\mathbf{W}^{(n)}, Q^{(n)})$, $n \geq 1$.

- We improve on the work of Dhara et al. [23], by weakening the conditions on the limit of encoding processes guaranteeing the convergence of the corresponding sequence of excursion length vectors.

- We show that the paths $t \mapsto \sum_{i=1}^m X^{(\infty)}_{k,i} \circ \gamma^{(\infty)}_i(t)$, where $X^{(\infty)}_{k,i}$ is the scaling limit of $X^{(n)}_{k,i}$, for all $k, i \in [m]$, satisfy the above conditions.

In Section 4, we introduce two explorations of the DCSBM. The first Exploration 4.1 is via the random field $X_{i,j}$ from (2.3), and the second one uses the random graph defining data. We

prove in Proposition 4.3 that these two explorations are equivalent, and soon after we prove Theorem 2.6.

In Section 5, we recall and expand on various elementary properties of the multi-dimensional first hitting times of [16] for deterministic fields $\mathbb{x}$. We then analyze these first hitting times whenever the field $\mathbb{x}$ is sufficiently smooth, and prove Theorem 2.8 in this special case. This analysis relies on several properties of homeomorphic inverses which fail in general.

In order to overcome this difficulty, in Section 6.1 we are lead to introducing a novel composition-like operator $\tilde{\circ}$, which has a remarkably good behaviour with respect to taking generalized inverses. For example, if g is an unbounded non-decreasing rcll function with left limits, and if g is strictly increasing at 0 in addition, then $g^{-1} \tilde{\circ}\, g = g \tilde{\circ}\, g^{-1} = \mathrm{id}$, where g^{-1} is the right-continuous generalized inverse of g. It is easy to see that the above identity fails in general (see also examples in Section 6.1)if $\tilde{\circ}$ is replaced by $\circ$.

Theorem 2.8 is proved in full generality in Section 6.2.

4 Breadth-first walk

Fix $\mathbf{W} = (\mathbf{w}^1, \mathbf{w}^2, \cdots, \mathbf{w}^m)$, a collection of m finite length vectors $\mathbf{w}^i \in \ell^2_{\downarrow}$. Furthermore fix an $m \times m$ symmetric matrix $Q = (Q_{i,j};\, i, j \in [m])$ with strictly positive entries along the diagonal and non-negative off-diagonal entries. All the processes considered in this section will depend on $\mathbf{W}$ and Q, but this will be mostly suppressed from the notation.

Let

$$\vec{R}_j = (R_{1,j}, \cdots, R_{m,j})^T \in \mathbb{R}^m_+, \tag{4.1}$$

so that the ith coordinate of $\vec{R}_j$ is $R_{i,j}$.

4.1 Two Explorations

The exploration we construct will involve m time-lines corresponding to m different coordinates (or types).

We will use the stochastic processes $X_{i,j}$ in (2.3) to construct the exploration. We keep track of two sequences of sets $(\mathcal{U}_k)_k$, $(\mathcal{U}^*_k)_k$ of *unexplored* vertices. The initial set $\mathcal{U}_0$ consists of all the vertices $\mathcal{U}_0 = \{(l, i) : l \leq \mathrm{len}(\mathbf{w}^i), i \in [m]\}$. At step k we maintain two stacks $\mathcal{A}_k$, $\mathcal{A}^*_k$ of *active* vertices. The initial stack $\mathcal{A}_0 = ()$ is empty. In addition we keep track of the set of *dead* vertices, which we denote by $\mathcal{D}_k$ in step k. A given vertex needs to be active in order to become dead in a later step, so $\mathcal{D}_0 = ()$ is empty as well. Recall that we write $[m]_{\vec{\rho}} = \{i \in [m] : \rho_i > 0\}$, and at step k denote by $\mathcal{U}^{\vec{\rho}}_k$ all the unexplored vertices $\mathcal{U}_k \cap \mathbb{N} \times [m]_{\vec{\rho}}$ with strictly positive $(Q, \vec{\rho})$-scaled mass.

We will inductively construct two sequences of m-dimensional stopping times $\vec{S}^*_k = (S^*_{k;1}, S^*_{k;2}, \cdots, S^*_{k;m})$ where $* \in \{L, R\}$, with respect to the filtration $\mathscr{F}(\vec{t})$ generated by $\mathbb{X}$. We set $\vec{S}^R_0 = \vec{0}$. Let ζ_k denote the number of connected components that have been discovered up-to step k. Note that $\zeta_0 = 0$. In the following exploration algorithm, we itemize the steps in order to facilitate the understanding of the subsequent proof.

Exploration 4.1 (Field Exploration)**.** Let $k = 1$.

(X0) **While** either $\mathcal{A}_{k-1} \neq ()$ or $\mathcal{U}^{\vec{\rho}}_{k-1} \neq \emptyset$ do as follows:

(X1) **Orientation:**

(a) On $\{\mathcal{A}_{k-1} = ()\} \cap \{\mathcal{U}^{\vec{\rho}}_{k-1} \neq \emptyset\}$ our exploration recorded all the vertices of the initial ζ_{k-1} components intersecting $[m]_{\vec{\rho}}$, and there are still some unexplored vertices left in $V(\mathcal{G}) \cap [m]_{\vec{\rho}}$. The algorithm increments $\zeta_k = \zeta_{k-1} + 1$, and defines

$$f(l, i; k) := \frac{\xi^i_l}{\rho_i Q_{i,i}} - \frac{S^R_{k-1;i}}{\rho_i}, \quad (l, i) \in \mathcal{U}^{\vec{\rho}}_{k-1},$$

and

$$\varpi(k) = (l_k, i_k) = \operatorname*{argmin}_{(l,i)\in\mathcal{U}_{k-1}} f(l, i; k), \qquad Y_{\zeta_k} = \min_{(l,i)\in\mathcal{U}_{k-1}} f(l, i; k).$$

Set $\mathcal{A}^*_{k-1} = (\varpi(k))$ and $\mathcal{U}^*_{k-1} = \mathcal{U}_{k-1} \setminus \{\varpi(k)\}$. We call $\varpi(k)$ the *root* of the ζ_kth component. For each $i \in [m]$, set $S^L_{k;i} = S^R_{k-1;i} + \rho_i Y_{\zeta_k}$, and let $\vec{S}^R_k = \vec{S}^L_k + w_{\varpi(k)} \vec{R}_{i_k}$, where here and below i_k is the above defined type of $\varpi(k)$. We define $N_k = k$ in this case (the reasons for this will be clear soon).

(b) Otherwise on $\{\mathcal{A}_{k-1} \neq ()\}$ we have (by induction)

$$\mathcal{A}_{k-1} = \Big(\varpi(k), \cdots, \varpi(N_k)\Big),$$

for some $N_k \geq k$. Set $\mathcal{A}^*_{k-1} := \mathcal{A}_{k-1}, \mathcal{U}^*_{k-1} = \mathcal{U}_{k-1}$, and $\zeta_k = \zeta_{k-1}$.

Note that $N_k - k + 1$ equals the length of the stack $\mathcal{A}^*_{k-1}$ almost surely.

(X2) **The (unexplored) neighbors of $\varpi(k)$:**

(a) The (newly discovered) neighbors of $\varpi(k)$ are the vertices

$$\mathcal{B}_k := \left\{(l, i) \in \mathcal{U}^*_{k-1} : \frac{\xi^i_l}{Q_{i,i}} \in [S^L_{k;i}, S^R_{k;i}).\right\}$$

Let $\chi(k) = \#\mathcal{B}_k$ be the cardinality of $\mathcal{B}_k$.

(b) On $\{\chi(k) = 0\}$ the algorithm jumps to (X3).

(c) Otherwise on $\{\chi(k) > 0\}$, we use the jump times $\xi^i_l/Q_{i,i} \in [S^L_{k;i}, S^R_{k,i})$ to order the elements of $\mathcal{B}_k$ as follows: Set

$$\varpi(N_k + 1), \cdots, \varpi(N_k + \chi(k)),$$

in the almost surely unique way such that $i \mapsto \mathbf{t}(\varpi(N_k + i))$ is non-decreasing on $\{1, \ldots, \chi(k)\}$, and such that ties are broken according to the rule $\{\mathbf{t}(\varpi(N_k + i)) = \mathbf{t}(\varpi(N_k + j)), i < j\} \subset \{\xi_{\varpi(N_k+i)} < \xi_{\varpi(N_k+j)}\}$, almost surely.

(d) For $j = 1, 2, \cdots \chi(k)$ write (temporarily) $\mathbf{t}_j = \mathbf{t}(\varpi(N_k + j))$, and define

$$\vec{S}^L_{N_k+j} = \vec{S}^R_{N_k+j-1} \qquad \text{and} \qquad \vec{S}^R_{N_k+j} = \vec{S}^L_{N_k+j} + w_{\varpi(N_k+j)} \vec{R}_{\mathbf{t}_j}.$$

(X3) **Update Sets and Stacks**: Set $M_k := N_k + \chi(k)$, and

on $\{M_k > k\}$ define

$$\mathcal{A}_k := \Big(\varpi(k + 1), \varpi(k + 2), \cdots, \varpi(N_k + \chi(k))\Big), \text{ while}$$

on $\{M_k = k\}$ define $\mathcal{A}_k := \emptyset$,

$$\mathcal{U}_k := \mathcal{U}^*_{k-1} \setminus \mathcal{B}_k,$$
$$\mathcal{D}_k := \mathcal{D}_{k-1} \cup \{\varpi(k)\}.$$

Increment k by 1 and go to step (X0).

We now describe another exploration algorithm, which is directly linked to the DCSBM graph. We use similar notation on purpose, except now every character will have an additional "widetilde" mark in the superscript. For example, we write $\widetilde{\mathcal{A}}_k$ to denote the analogue of $\mathcal{A}_k$, and we define $\widetilde{\zeta}_0 = 0$. Moreover, all the initial values of stacks $\widetilde{\mathcal{U}}_0, \widetilde{\mathcal{A}}_0, \widetilde{\mathcal{D}}_0$ are equal to the values of their corresponding analogues in Exploration 4.1.

We will use several times in the sequel the following notation. Let $\mathcal{F}$ be a σ-field, and let $W \geq 0$ be a non-negative $\mathcal{F}$-measurable random variable. Then

$$X|\mathcal{F} \sim \mathrm{Exp}(W),$$

means that the conditional law of X given $\mathcal{F}$ is exponential with rate W.

Exploration 4.2 (Graph Exploration)**.** Let $k = 1$.

(G0) **While** either $\widetilde{\mathcal{A}}_{k-1} \neq ()$ or $\widetilde{\mathcal{U}}^{\vec{\rho}}_{k-1} \neq \emptyset$ do as follows:

(G1) **Orientation:**

(a) On $\{\widetilde{\mathcal{A}}_{k-1} = ()\} \cap \{\widetilde{\mathcal{U}}^{\vec{\rho}}_{k-1} \neq \emptyset\}$ our exploration recorded all the vertices of the initial $\widetilde{\zeta}_{k-1}$ components of $\mathcal{G}$ which intersect $[m]_{\vec{\rho}}$, and the unexplored part of $\mathcal{G}$ non-trivially intersects $[m]_{\vec{\rho}}$. The algorithm increments $\widetilde{\zeta}_k = \widetilde{\zeta}_{k-1} + 1$, and samples $\widetilde{\varpi}(k) = (\widetilde{l}_k, \widetilde{i}_k)$ from $\mathcal{G} \setminus \mathcal{D}_{k-1}$ according to

$$\mathbb{P}\Big(\widetilde{\varpi}(k) = (l,i)\big|\mathcal{H}_k\Big) \propto \rho_i Q_{i,i} w^i_l 1_{[(l,i)\in\widetilde{\mathcal{U}}_{k-1}]},$$

$$\widetilde{Y}_{\widetilde{\zeta}_k}|\mathcal{H}_k \sim \mathrm{Exp}\left(\sum_{(l,i)\in\mathcal{G}\setminus\mathcal{D}_{k-1}} \rho_i Q_{i,i} w^i_l\right),$$

where $\mathcal{H}_k$ is the σ-field generated by the first $k-1$ steps of the algorithm, and where $\widetilde{Y}_{\widetilde{\zeta}_k}$ and $\widetilde{\varpi}(k)$ are conditionally independent given $\mathcal{H}_k$. Set $\widetilde{\mathcal{A}}^*_{k-1} = (\widetilde{\varpi}(k))$ and $\widetilde{\mathcal{U}}^*_{k-1} = \widetilde{\mathcal{U}}_{k-1} \setminus \{\widetilde{\varpi}(k)\}$. Call $\widetilde{\varpi}(k)$ the *root* of the $\widetilde{\zeta}_k$th component of $\mathcal{G}$, and define $\widetilde{N}_k = k$.

(b) Otherwise on $\{\widetilde{\mathcal{A}}_{k-1} \neq ()\}$ we have (by induction)

$$\widetilde{\mathcal{A}}_{k-1} = \Big(\widetilde{\varpi}(k), \cdots, \widetilde{\varpi}(\widetilde{N}_k)\Big),$$

for some $\widetilde{N}_k \geq k$. Set $\widetilde{\mathcal{A}}^*_{k-1} := \widetilde{\mathcal{A}}_{k-1}, \widetilde{\mathcal{U}}^*_{k-1} = \widetilde{\mathcal{U}}_{k-1}$, and $\widetilde{\zeta}_k = \widetilde{\zeta}_{k-1}$.

Note that $\widetilde{N}_k - k + 1$ equals the length of the stack $\widetilde{\mathcal{A}}^*_{k-1}$ almost surely.

(G2) **The (unexplored) neighbors of** $\widetilde{\varpi}(k)$:

(a) Let $\widetilde{\mathcal{B}}_k$ be the neighbors of $\widetilde{\varpi}(k)$ contained in $\widetilde{\mathcal{U}}^*_{k-1}$. We set $\widetilde{\chi}(k) = \#\widetilde{\mathcal{B}}_k$ and call it the number of children of $\widetilde{\varpi}(k)$.

(b) On $\{\widetilde{\chi}(k) = 0\}$ the algorithm jumps to (G3).

(c) Otherwise on $\{\widetilde{\chi}(k) > 0\}$ the elements of $\widetilde{\mathcal{B}}_k$ are ordered as

$$\widetilde{\varpi}(\widetilde{N}_k + 1), \cdots, \widetilde{\varpi}(\widetilde{N}_k + \widetilde{\chi}(k)),$$

first non-decreasingly with respect to their type, and for each $i \in [m]$ using conditionally and mutually independent (of all the information which our exploration collected up to this point) auxiliary size-biasing (with respect to weight) of elements of type i.

(G3) **Update Sets and Stacks**: Set $\widetilde{M}_k := \widetilde{N}_k + \widetilde{\chi}(k)$, and

on $\{\widetilde{M}_k > k\}$ define

$$\widetilde{\mathcal{A}}_k := \Big(\widetilde{\varpi}(k+1), \widetilde{\varpi}(k+2), \cdots, \widetilde{\varpi}(\widetilde{N}_k + \widetilde{\chi}(k))\Big), \text{ while}$$

on $\{\widetilde{M}_k = k\}$ define $\widetilde{\mathcal{A}}_k := \emptyset$,

$$\widetilde{\mathcal{U}}_k := \widetilde{\mathcal{U}}^*_{k-1} \setminus \widetilde{\mathcal{B}}_k$$
$$\widetilde{\mathcal{D}}_k := \widetilde{\mathcal{D}}_{k-1} \cup \{\widetilde{\varpi}(k)\}.$$

Increment k by 1 and go to step (G0).

Define $\zeta_\infty := \sup_k \zeta_k$ and $\widetilde{\zeta}_\infty := \sup_k \widetilde{\zeta}_k$. Note that both ζ_∞ and $\widetilde{\zeta}_\infty$ are finite random variables, and more importantly that the numbers of steps in the above while loops, respectively

$$K := \inf\{k : \zeta_k = \zeta_\infty\} \text{ and } \widetilde{K} := \inf\{k : \widetilde{\zeta}_k = \widetilde{\zeta}_\infty\},$$

are finite almost surely. Our key proposition is stated next.

Proposition 4.3. *Explorations 4.1 and 4.2 are equal in law. More precisely,*

$$(\mathcal{A}_k, \mathcal{A}^*_k, \mathcal{U}_k, \mathcal{U}^*_k, \zeta_k, Y_{\zeta_k}, \mathcal{B}_k, N_k : k \le K) \stackrel{d}{=} (\widetilde{\mathcal{A}}_k, \widetilde{\mathcal{A}}^*_k, \widetilde{\mathcal{U}}_k, \widetilde{\mathcal{U}}^*_k, \widetilde{\zeta}_k, \widetilde{Y}_{\widetilde{\zeta}_k}, \widetilde{\mathcal{B}}_k, \widetilde{N}_k : k \le \widetilde{K}), \quad (4.2)$$

and $(\varpi(k) : k \le K) \stackrel{d}{=} (\widetilde{\varpi}(k) : k \le \widetilde{K})$ *in particular.*

We delay the proof until Section 4.3.

4.2 Preliminary Lemmas

Denote by $\le$ (typically we write $\vec{s} \le \vec{t}$) the standard coordinate-wise comparison partial order on $(-\infty, \infty]^m$. In complete analogy with the one-dimensional time setting, we can now define filtrations (and related notions) indexed by $\mathbb{R}^m_+ \subset (-\infty, \infty]^m$.The reader is referred to [26, Section 2.8] for concepts and results which we will typically use in the sequel without further mention. In particular, the filtration $\mathscr{F} = (\mathscr{F}(\vec{t}); \vec{t} \in \mathbb{R}^m_+)$ of our field $\mathbb{X}$ is defined by the standard completion of $\sigma\left\{\mathbb{X}(\vec{s}) : \vec{s} \le \vec{t}\right\}$. It is clear that $\mathbb{X}$ is adapted to $\mathscr{F}$. Furthermore it is easy to see that $\mathbb{X}$ has right-continuous paths with respect to (the above m-dimensional partial order in time and) the Euclidean topology on the state space $\mathbb{R}^m$. We conclude that $\mathbb{X}$ is $\mathscr{F}$-progressively measurable.

We will need the following analogue of the m-dimensional strong Markov property.

Lemma 4.4. *Let $\vec{S}$ be a $\mathscr{F}$-stopping time such that $\mathbb{P}(\vec{S} < \infty) = 1$. Then*

$$\left(\mathbb{X}(\vec{t} + \vec{S}) - \mathbb{X}(\vec{S}); \vec{t} \in \mathbb{R}^m_+\right) \Big| \mathscr{F}(\vec{S}) \stackrel{d}{=} \left(\tilde{\mathbb{X}}(\vec{t}); \vec{t} \in \mathbb{R}^m_+\right), \quad (4.3)$$

where $\tilde{\mathbb{X}}$ is the random field distributed as $\mathbb{X}^{\tilde{\mathbf{W}},Q}$ with a random (and $\mathscr{F}(\vec{S})$-measurable) collection of weights $\tilde{\mathbf{W}} = (\tilde{\mathbf{w}}^1, \cdots, \tilde{\mathbf{w}}^m)$, such that $\tilde{\mathbf{w}}^j$ is the unique vector in $\ell^2_\downarrow$ of finite length whose entries are the non-decreasingly ordered elements of $\{w^j_l : \frac{1}{Q_{j,j}}\xi^j_l > S_j\}$.

Proof. The assumption is that $\vec{S}$ is a non-negative random vector such that $\{\vec{S} \leq \vec{t}\} \in \mathscr{F}(\vec{t})$ for all $\vec{t}$. The σ-field $\mathscr{F}(\vec{S})$ consists of all measurable A such that $A \cap \{\vec{S} \leq \vec{s}\} \in \mathscr{F}(\vec{s})$ for all $\vec{s} \geq \vec{0}$.

As in the 1-dimensional setting, we have $(\vec{S}, \mathbb{X}(\vec{S})) \in \mathscr{F}(\vec{S})$. We know that the field of increments $\mathbb{X}(\cdot + \vec{S}) - \mathbb{X}(\vec{S})$ depends deterministically on the residual exponential random variables $(\xi_l^j - Q_{j,j}S_j)_{j\in[m],l\geq 1,w_l^j>0,\xi_l^j/Q_{j,j}>S_j}$ in the same way that the original field $\mathbb{X}$ depends on $(\xi_l^j)_{j\in[m],l\geq 1,w_l^j>0}$. We need to check that, given $\mathscr{F}(\vec{S})$, $(\xi_l^j - Q_{j,j}S_j)_{j\in[m],l\geq 1,w_l^j>0,\xi_l^j/Q_{j,j}>S_j}$ are again independent exponentials, where $\xi_l^j - Q_{j,j}S_j$ has exponential (rate w_l^j) distribution. For this we first note that $\{\xi_l^j/Q_{j,j} > S_j\} \in \mathscr{F}(\vec{S})$, for each $l \geq 1$ and $j \in [m]$ such that $w_l^j > 0$, and moreover that $\mathscr{F}(\vec{S})$ is in fact generated by $\vec{S}$ and the family of exponentials $(\xi_l^k/Q_{k,k})_{k\in[m],l\geq 1,w_l^k>0,\xi_l^k/Q_{k,k}\leq S_j}$ which occur prior to $\vec{S}$. Therefore, on $\{\xi_l^j/Q_{j,j} > S_j\}$, we have

$$\mathbb{P}\left(\frac{\xi_l^j}{Q_{j,j}} - S_j > u \middle| \mathscr{F}(\vec{S})\right) = \mathbb{P}\left(\frac{\xi_l^j}{Q_{j,j}} - S_j > u \middle| \vec{S}, \left(\frac{\xi_l^k}{Q_{k,k}}\right)_{k\in[m],l\geq 1,w_l^k>0,\xi_l^k/Q_{k,k}\leq S_j}\right). \tag{4.4}$$

Due to the independence of the original family $(\xi_l^j)_{j\in[m],l\geq 1,w_l^j>0}$ of exponentials, it is now particularly easy to check that the RHS in (4.4) equals $e^{-uQ_{j,j}w_l^j}$ almost surely on $\{\xi_l^j/Q_{j,j} > S_j\}$, provided that the stopping time $\vec{S}$ is a discrete random vector.

The previous paragraph can be generalized in an obvious way in order to prove conditional independence (given $\mathscr{F}(\vec{S})$) of residual quantities $\frac{\xi_l^j}{Q_{j,j}} - S_j$ over all relevant l and j. Together with the above discussion, this confirms (4.3) in the discrete $\vec{S}$ setting. Deriving

$$\mathbb{P}\left(\frac{\xi_l^j}{Q_{j,j}} - S_j > u \middle| \mathscr{F}(\vec{S})\right) = e^{-uQ_{j,j}w_l^j} \text{ a.s.} \tag{4.5}$$

and its joint distribution counterparts, or equivalently, deriving (4.3) for any stopping time $\vec{S}$ is done in a standard way (by approximating $\vec{S}$ from above with a sequence of discrete stopping times). □

The following corollary is immediate and will also be used frequently in the sequel.

Corollary 4.5. *Let $\vec{S}$ be an $\mathscr{F}$-stopping time such that $\mathbb{P}(\vec{S} < \infty) = 1$. Then for any $\vec{t} \in (0,\infty)^m$, and all $i, j \in [m]$*

$$\mathbb{P}(X_{i,j}(t_j + S_j) = X_{i,j}((t_j + S_j)-)) = 1.$$

More generally, suppose that $\vec{U} \in \mathbb{R}_+^m$ is an $\mathscr{F}(\vec{S})$-measurable random variable such that $\bigcup_j \{X_{j,j}(S_j) \neq X_{j,j}(S_j-), U_j = 0\}$ is an event of probability zero. Then, for all $i, j \in [m]$

$$\mathbb{P}(X_{i,j}(U_j + S_j) = X_{i,j}((U_j + S_j)-)) = 1.$$

The following lemma is also easy.

Lemma 4.6. *Suppose that $\vec{S}$ is an almost surely finite $\mathscr{F}$-stopping time, $\vec{U} \in \mathscr{F}(\vec{S})$ is a random variable such that $\mathbb{P}(\vec{U} \geq \vec{S}) = 1$. Let (L, I) be an $\mathscr{F}(\vec{S})$-measurable random index. Then $\vec{U}$ and $\vec{U} + w_L^I \vec{R}_I 1_{\{\xi_L^I/Q_{I,I}\leq S_I\}}$ are also $\mathscr{F}$-stopping times.*

The above lemma implies that in (X2)(c) all the random vectors $\vec{S}^L_{N_k+j}$, where $* \in \{L, R\}$, are stopping times provided that both $\vec{S}^L_{N_k}$ and $\vec{S}^R_{N_k}$ are stopping times. The proof of the next lemma is again a consequence of Lemma 4.4.

Lemma 4.7. *Let $\vec{S}$ be an $\mathscr{F}$-stopping time, and let $\mathcal{U} = \{(l,i) : \xi^i_l/Q_{i,i} > S_i\}$. Define*

$$Z = \min_{(l,i)\in\mathcal{U}} \left\{ \frac{1}{\rho_i} \left(\frac{\xi^i_l}{Q_{i,i}} - S_i \right) \right\}.$$

Then $\vec{S} + Z\vec{\rho}$ is an $\mathscr{F}$-stopping time, and furthermore

$$Z \big| \mathscr{F}(\vec{S}) \sim \mathrm{Exp}\left(\sum_{(l,i)\in\mathcal{U}} \rho_i Q_{i,i} w^i_l \right),$$

and

$$\mathbb{P}\left(\operatorname*{argmin}_{(l,i)\in\mathcal{U}} \left\{ \frac{1}{\rho_i} \left(\frac{\xi^i_l}{Q_{i,i}} - S_i \right) \right\} = (l_0, i_0) \big| \mathscr{F}(\vec{S}) \right) 1_{[(l_0,i_0)\in\mathcal{U}]} \propto \rho_{i_0} Q_{i_0,i_0} w^{i_0}_{l_0} 1_{[(l_0,i_0)\in\mathcal{U}]}.$$

Proof. As already noted, we will apply Lemma 4.4. More precisely, it suffices to prove the statements on the conditional law of Z and $\operatorname{argmin}_{(l,i)\in\mathcal{U}} \left\{ \frac{1}{\rho_i} \left(\frac{\xi^i_l}{Q_{i,i}} - S_i \right) \right\}$ in the setting where $\mathbb{P}(\vec{S} = \vec{0}) = 1$. Note that these are a clear consequence of the elementary properties of (conditionally) independent exponential random variables.

Furthermore, $Y_1\vec{\rho}$ (where Y_1 is defined in (X1)(a)) is an $\mathscr{F}$-stopping time since

$$\begin{aligned} \{Y_1\vec{\rho} \le \vec{t}\} &= \left\{ \exists (l,i) : \frac{\xi^i_l}{\rho_i Q_{i,i}} \rho_i \le t_i \right\} = \left\{ \exists (l,i) : \frac{\xi^i_l}{Q_{i,i}} \le t_i \right\} \\ &= \{\exists i : X_{i,i} \text{ has a jump in } [0, t_i]\}, \end{aligned}$$

and the event on the RHS is clearly $\mathscr{F}(\vec{t})$-measurable. Applying Lemma 4.4, we arrive to the conclusion that $Z\vec{\rho}$ is a stopping-time with respect to the natural filtration of $\mathbb{X}(\cdot + \vec{S}) - \mathbb{X}(\vec{S})$, and a little thought is needed to see that this is enough to conclude that $\vec{S} + Z\vec{\rho}$ is a stopping time with respect to $(\mathscr{F}(\vec{t}), \vec{t} \ge \vec{0})$. □

By induction, Lemmas 4.6 and 4.7 imply the following:

Lemma 4.8. *Recall Exploration 4.1. The vectors $\vec{S}^L_k$ and $\vec{S}^R_k$ constructed in (X1)(a) are $\mathscr{F}$-stopping times, for any relevant k.*

Remark 4.9. Lemma 4.7 guarantees that the start of exploration (in (X1)(a)) of each connected component occurs at a stopping times. Lemma 4.6 implies that $\vec{S}^L_k$ and $\vec{S}^R_k$ are stopping times for each k such that $\varpi(k)$ is not a root of a connected component (or equivalently, $\varpi(k)$ is discovered in (X2)(c) at some earlier step).

The following result will be used to prove the required "equivalence" of (X2) in Exploration 4.1 and (G2) in Exploration 4.2.

Lemma 4.10. *Suppose that $\vec{S}$ is an $\mathscr{F}$-stopping time. Let $\mathcal{U}$ be the $\mathscr{F}(\vec{S})$-measurable collection defined as $\{(l,i) : i \in [m], l \in \mathrm{len}(\mathbf{w}^i), \xi^i_l/Q_{i,i} > S_i\}$. Let (L, I) be an $\mathscr{F}(\vec{S})$-measurable random index.*

Then

$$\mathbb{P}\left(\frac{\xi^j_r}{Q_{j,j}} \in [S_j, S_j + w^I_L R_{j,I}) \Big| \mathscr{F}(\vec{S}) \right) 1_{[(r,j)\in\mathcal{U},(L,I)\notin\mathcal{U}]} = \left(1 - e^{-Q_{j,I} w^I_L w^j_r} \right) 1_{[(r,j)\in\mathcal{U},(L,I)\notin\mathcal{U}]}.$$

Proof. Due to Lemma 4.4, we can suppose without loss of generality that $\vec{S} = \vec{0}$, where we replace $\mathbf{W}$ with $\widetilde{\mathbf{W}}$.

Since $\xi_r^j \sim \mathrm{Exp}(w_r^j)$, we have

$$\mathbb{P}\left(\xi_r^j / Q_{j,j} \in [0, w_L^I R_{j,I})\right) = 1 - \exp\left(-Q_{j,j} R_{j,I} w_L^I w_r^j\right).$$

The claim now follows from the identities $R_{j,I} Q_{j,j} \equiv Q_{j,I}$ (see (2.2)). □

4.3 Proof of Proposition 4.3

The argument relies on the inductive coupling of the two explorations. We will need the following notation. Let $\mathcal{E}_k$ (resp. $\widetilde{\mathcal{E}}_k$) denote the σ-algebra generated by the initial $k \wedge K$ (resp. $k \wedge \widetilde{K}$) steps of Exploration 4.1 (resp. 4.2). We begin by noting that by induction

$$\mathcal{E}_k = \mathscr{F}\left(\vec{S}^R_{k \wedge K}\right). \tag{4.6}$$

Indeed, note that the algorithm gathers all the information in $\mathcal{E}_k$ while passing through steps (X1)–(X3) in the initial $k \wedge K$ rounds (in fact, this information is gathered already by the end of (X2)(c) in round $k \wedge K$). In particular, all the jump times of $\mathbb{X}$ occurring before $\vec{S}^R_{k \wedge K}$ are included in $\mathcal{E}_k$, hence $\mathcal{E}_k \supset \mathscr{F}(\vec{S}^R_{k \wedge K})$. In the opposite direction, note that using $\mathscr{F}(\vec{S}^R_{k \wedge K})$ we can reconstruct the field exploration for the first $k \wedge K$ steps (or rounds), so that $\mathcal{E}_k \subset \mathscr{F}(\vec{S}^R_{k \wedge K})$.

Unless needed for additional clarity, we will abuse notation and write "step k" instead of "step $k \wedge K$" (or "step $k \wedge \widetilde{K}$"). We will also write $\vec{S}^*_k$ instead of $\vec{S}^*_{k \wedge K}$, where as usual $* \in \{L, R\}$.

4.3.1 The base of induction

If $k = 1$, we need to compare steps (X1)(a) and (G1)(a). In this case, Lemma 4.7 (with $\vec{S} = \vec{0}$) implies that the respective outputs of steps (X1)(a) and (G1)(a) are equal in distribution. We couple them so that they are equal almost surely.

Recall that $\varpi(1) = (l_1, i_1)$ and similarly $\widetilde{\varpi}(1) = (\widetilde{l}_1, \widetilde{i}_1)$. As just explained, in our coupling $(\widetilde{l}_1, \widetilde{i}_1) = (l_1, i_1)$. Continuing onto (X2) and (G2), for each vertex $v \in \mathcal{U}_0^* = \widetilde{\mathcal{U}}_0^* = \mathcal{U}_0 \setminus \{\varpi(1)\}$

$$\mathbb{P}(v \in \mathcal{B}_1 \big| \mathcal{E}_0, \mathcal{U}_0^*) = 1 - \exp\left(-Q_{i_1, \mathbf{t}(v)} w_v w_{l_1}^{i_1}\right),$$

due to Lemma 4.10, and similarly,

$$\mathbb{P}\left(v \in \widetilde{\mathcal{B}}_1 \big| \widetilde{\mathcal{E}}_0, \widetilde{\mathcal{U}}_0^*\right) = 1 - \exp\left(-Q_{i_1, \mathbf{t}(v)} w_v w_{l_1}^{i_1}\right),$$

due to the very definition of $\mathcal{G}$ (in particular, the independence of edge connections in DCSBM). Therefore, $\mathcal{B}_1 | (\mathcal{E}_0, \mathcal{U}_0^*) \overset{d}{=} \widetilde{\mathcal{B}}_1 | (\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{U}}_0^*)$, and more importantly $\xi_l^i / Q_{i,i} \in [S^L_{1;i}, S^R_{1,i})$ can be used (verbatim) to generate the output of (G2)(c). More precisely, we repeatedly apply Lemma 4.4 and Lemma 4.8 to see that the residual clocks in Step (c) of (X2) are still (conditionally) independent exponential random variables. There is no analogue of (X2)(d) in Exploration 4.2. In particular, the sequences $(\vec{S}^*_k)_k$ do not appear in the statement of Proposition 4.3, yet they play an important role in its proof. As before, we couple the outputs of (X2)(c) and (G2)(c) so that they are equal almost surely.

Since (X3) (resp. (G3)) depends deterministically on the output of (X2) (resp. (G2)), we conclude that (X3) and (G3) are clearly almost surely equal (at least, if $k = 1$) in our coupling.

4.3.2 Induction Step

Suppose that on $\{K \geq k-1\} = \{\widetilde{K} \geq k-1\}$ we already have a coupling in which the outputs of the two explorations are identical almost surely in steps $1, 2, \cdots, k-1$. We need to extend it so that $\{K = k-1\} = \{\widetilde{K} = k-1\}$, and furthermore that on $\{K \geq k\} = \{\widetilde{K} \geq k\}$ the outputs of the two explorations in step k are again identical almost surely.

The induction step is essentially identical to the base case $k = 1$, with additional applications of Lemma 4.8 whenever necessary. Here we only sketch the argument.

Recall that the two explorations have the same condition for entering the "while loop" in steps (X0) and (G0), respectively.

On $\{K \geq k-1\} \cap \{\mathcal{A}_{k-1} \neq \emptyset\} = \{\widetilde{K} \geq k-1\} \cap \{\widetilde{\mathcal{A}}_{k-1} \neq \emptyset\}$ the outputs of (X1) and (G1) are a deterministic function of the outputs of step $k-1$ in the already constructed coupling, and therefore they are identical almost surely. On $\{K \geq k-1\} \cap \{\mathcal{A}_{k-1} = \emptyset\} = \{\widetilde{K} \geq k-1\} \cap \widetilde{\mathcal{A}}_{k-1} = \emptyset\}$ we rely on $\mathcal{U}_{k-1} = \widetilde{\mathcal{U}}_{k-1}$ and therefore $\mathcal{U}_{k-1}^{\vec{\rho}} = \widetilde{\mathcal{U}}_{k-1}^{\vec{\rho}}$ almost surely. Hence $\{K = k-1\} =$

$$\{K \geq k-1, \mathcal{A}_{k-1} = \emptyset, \mathcal{U}_{k-1}^{\vec{\rho}} = \emptyset\} = \{\widetilde{K} \geq k-1, \widetilde{\mathcal{A}}_{k-1} = \emptyset, \widetilde{\mathcal{U}}_{k-1}^{\vec{\rho}} = \emptyset\} = \{\widetilde{K} = k-1\},$$

up to null-sets, and furthermore on $\{K > k-1\} \overset{a.s.}{=} \{\widetilde{K} > k-1\}$ we apply (as in the base of induction) Lemmas 4.4, 4.7 and 4.8 in order to couple the outputs of (X1)(a) and (G1)(a).

The coupling in steps (X2) and (G2), and in steps (X3) and (G3), respectively, is done as in the base case. This concludes the proof of Proposition 4.3.

4.4 Exploration 4.1 and hitting times T

Recall the definition of $X_{i,j}$ in (2.3). Due to elementary properties of independent exponentials, for each fixed j, there is an almost surely uniquely defined permutation π^j on $\text{len}(\mathbf{w}^j)$ letters such that

$$\xi^j_{\pi^j(1)} < \xi^j_{\pi^j(2)} < \cdots < \xi^j_{\pi^j(\text{len}(\mathbf{w}^j))}. \tag{4.7}$$

We will sometimes write $\pi(l, j)$ in place of the value $\pi^j(l)$ of the permutation π^j at l. We shall also sometimes write $\pi(l, j)$ instead of vertex $(\pi^j(l), j)$.

As in the proof of Lemma 4.7, observe that with probability 1

$$\mathbb{X}(\vec{t}-) = -\vec{t} \quad \text{if and only if} \quad \vec{t} \leq \left(\frac{1}{Q_{1,1}}\xi_{\pi(1,1)}, \cdots, \frac{1}{Q_{m,m}}\xi_{\pi(1,m)}\right) \tag{4.8}$$

since the first jump time on the jth time-line is $\frac{1}{Q_{j,j}}\xi_v$, where $v = \pi(1, j)$, and since for all $\vec{t}$ with jth component $t_j \geq \frac{1}{Q_{j,j}}\xi_{\pi(1,j)}$ we have by definition $X_j(\vec{t}) \geq X_{j,j}(t_j) \geq -t_j + w_{\pi(1,j)} > -t_j$.

Recall the definition of Y_1 in (X1)(a) and recall that $Y_1\vec{\rho}$ is an $\mathscr{F}$-stopping time (see Lemma 4.7). Due to (4.8) we have that

$$\{T(y) = y\vec{\rho}\} = \{Y_1 \geq y\} \text{ a.s..} \tag{4.9}$$

The above remarks together with standard properties of exponentials imply the following.

Lemma 4.11. $\mathbf{T}(y) = y\vec{\rho}$ *for* $y \leq Y_1$ *and* $\mathbf{T}(Y_1+) \neq \mathbf{T}(Y_1)$.

Now let $\varpi(1), \varpi(2), \cdots, \varpi(N)$ be the vertices of the first connected component discovered in Exploration 4.1.

More precisely, define $N := \inf\{k : \zeta_k = 2\} - 1$. In the statement of the next result we use the above notation.

Lemma 4.12. *We have*

$$\mathbf{T}(Y_1) = \vec{S}_1^L \qquad \text{and} \qquad \mathbf{T}(Y_1+) = \vec{S}_N^R \qquad \text{almost surely.} \tag{4.10}$$

On $\{K > N\}$ *the restriction of* $\mathbf{T}$ *to* $(Y_1, Y_1 + Y_2]$ *is an affine function*

$$y \mapsto \mathbf{T}(y) = \mathbf{T}(Y_1+) + (y - Y_1)\vec{\rho}.$$

More generally, for any $l \leq r$ *and* p *positive integers, on the event* $\{\zeta_{l-1} = p-1, \zeta_l = p, \zeta_r = p\} \cap (\{K = r\} \cup \{K > r, \zeta_{r+1} = p+1\})$ *(where* $\{\varpi(l), \cdots, \varpi(r)\}$ *is the* p^{th} *connected component explored), we have*

$$\mathbf{T}\left(\sum_{n=1}^{p} Y_n\right) = \vec{S}_l^L \qquad \text{and} \qquad \mathbf{T}\left(\sum_{n=1}^{p} Y_n+\right) = \vec{S}_r^R \qquad \text{almost surely, and} \tag{4.11}$$

on the above event intersected with $\{K > r\}$ *the restriction of* $\mathbf{T}$ *to* $(\sum_{n=1}^{p} Y_n, \sum_{n=1}^{p+1} Y_n]$ *is an affine function*

$$y \mapsto \mathbf{T}(y) = \mathbf{T}\left(\sum_{n=1}^{p} Y_n+\right) + \left(y - \sum_{n=1}^{p} Y_n\right)\vec{\rho}.$$

Proof. We prove in detail the statements which concern the first encountered component, or more precisely the process $\mathbf{T}$ before and after its first jump. The statements related to the pth jump of $\mathbf{T}$ can be proven in a similar fashion, using the strong Markov property of $\mathbb{X}$. We provide a sketch, and leave the details to an interested reader.

The first identity in (4.10) is an easy consequence of the definition of $\vec{S}_1^L$ in (X1)(a).

In order to establish the second identity in (4.10) it suffices to show that almost surely

$$\mathbf{T}(Y_1+) \leq S_N^R + \delta\vec{\rho}, \text{ for any } \delta > 0, \text{ and} \tag{4.12}$$

$$\mathbf{T}(Y_1 + \varepsilon) > S_N^R, \text{ for any } \varepsilon > 0. \tag{4.13}$$

As a preliminary calculation we show (see Corollary 4.5) that almost surely

$$X_j(S_{N;i}^R-) = X_j(S_{N;i}^R) = \sum_{j=1}^{m} X_{j,i}(S_{N;i}^R) = -\rho_j Y_1, \text{for each } j \in [m]. \tag{4.14}$$

Let $\vec{e}_k$ be the kth standard basis vector of $\mathbb{R}^m$, and note that by linearity

$$\vec{S}_N^R - \vec{S}_1^L = \sum_{p=1}^{N} \vec{R}_{\mathbf{t}(\varpi(p))} w_{\varpi(p)} = R\left(\sum_{p=1}^{N} w_{\varpi(p)} \vec{e}_{\mathbf{t}(\varpi(p))}\right) = R\vec{\mathscr{M}}, \tag{4.15}$$

where $\vec{\mathscr{M}}$ is the total weight vector (broken by type, see (2.8)) of the first explored component.

In addition, for $i \neq j$ we have by construction of the stopping time $\vec{S}_N^R$ (see again (X2)(d) and (2.3)) that

$$X_{j,i}(S_{N;i}^R) = \sum_{\substack{p:\mathbf{t}(\varpi(p))=i \\ p \leq N}} R_{j,i} w_{\varpi(p)}$$

as each jump of $X_{j,i}$ that occurs before time $S_{N;i}^R$ corresponds either to $\varpi(1)$ or to a (type i) child of some vertex from $\varpi(1), \cdots, \varpi(N)$. Similarly,

$$X_{j,j}(S_{N;j}^R) = -S_{N;j}^R + \sum_{\substack{p:\mathbf{t}(\varpi(p))=j \\ p \leq N}} w_{\varpi(p)}.$$

Adding up the identities in the last two displays and recalling that $R_{j,j} = 1$ (see (2.2)) implies

$$\sum_{i=1}^{m} X_{j,i}(S^R_{N;i}) = -S^R_{N;j} + \sum_{i=1}^{m} \sum_{\substack{p:\mathbf{t}(\varpi(p))=i \\ p\le N}} R_{j,i} w_{\varpi(p)} = -S^R_{N;j} + \sum_{p=1}^{N} R_{j,\mathbf{t}(\varpi(p))} w_{\varpi(p)}$$
$$= -S^R_{N;j} + \vec{e}_j^{\,T} R \vec{\mathscr{M}} = -S^L_{1;j} = -\rho_j Y_1,$$

where we used (4.15) for the second to last identity.

We now know that $\mathbb{X}(\vec{S}^R_N) = -Y_1\vec{\rho}$, and furthermore recall that the stack of active vertices (of the first explored component) is exhausted at time $\vec{S}^R_N$. The field $\mathbb{X}$ evolves deterministically for $t_l \in (S^R_{N;l}, S^R_{N;l}+Y_2\rho_l)$ where Y_2 is again defined in (X1)(a), and where $\vec{S}^L_{N+1} := \vec{S}^R_N + Y_2\vec{\rho}$ on $\{Y_2 < \infty\}$. More precisely, during $(\vec{S}^R_{N;l}, \vec{S}^R_{N;l} + Y_2\rho_l)$, the lth coordinate of $\mathbb{X}$ deterministically decreases at rate 1. If $\rho_l = 0$ the lth coordinate of $\mathbb{X}$ is not relevant in (X1)(a). In particular, $\{\delta < Y_2\} = \{\mathbb{X}(\vec{S}^R_N + \delta\vec{\rho}) = -(\delta + Y_1)\rho\}$ and therefore we have

$$\{\delta < Y_2\} \subset \{\mathbf{T}(Y_1 + \delta) \le \vec{S}^R_N + \delta\vec{\rho}\}, \tag{4.16}$$

which is enough for concluding (4.12). Note that $\vec{S}^L_{N+1}$ is defined only on $\{Y_2 < \infty\}$, but the rest of the argument is also valid on $\{Y_2 = \infty\}$, the event that the exploration process ends upon exploring the first (and only) connected component which intersects $[m]_{\vec{\rho}}$.

We next show (4.13). Again due to Corollary 4.5 (or by the reasoning in the previous paragraph) we know that $\mathbf{T}(Y_1 + \varepsilon)$ is a point of continuity for $\mathbb{X}$. We will show by induction that $\mathbf{T}(Y_1 + \varepsilon) \ge \vec{S}^R_k$ for all $k = 1, 2, \cdots, N$.

To do this, we observe that for each $p = 2, \cdots, N$ the vertex $\varpi(p) = (l_p, i_p)$ is discovered strictly before the start of its corresponding observation window in (X2)(a). In terms of the exponential clock $\xi^{i_p}_{l_p}$, this property can be written as

$$\frac{\xi^{i_p}_{l_p}}{Q_{i_p,i_p}} < S^R_{p-1;i_p} = S^L_{p;i_p}, \quad p \ge 2,$$

where the last identity above follows form (X2)(d). The root vertex $\varpi(1)$ is discovered at S^L_{1,i_1}, the start of its observation window.

Therefore, on $\{k \le N \le K\}$, before time $\vec{S}^L_k$ there are at least $\varpi(1), \ldots, \varpi(k)$ already discovered in Exploration 4.1. So for each $k = 1, 2, \cdots, N$ and $i \ne j$ we have

$$X_{j,i}(t_i) \ge \sum_{p\le k: i_p = i} R_{j,i} w_{\varpi(p)}, \ \forall t_i \ge S^L_{k;i}, \text{ and moreover} \tag{4.17}$$

$$X_{j,j}(t_j) \ge -S^R_{k;j} + \sum_{p\le k: i_p = j} w_{\varpi(p)}, \ \forall t_j \in [S^L_{k;j}, S^R_{k;j}], \text{ and} \tag{4.18}$$

$$X_{j,j}(t_j) \ge -s_j, \ \forall t_j \le s_j. \tag{4.19}$$

Similarly, if $k = 1$ our exploration algorithm guarantees that for $j \ne i_1$

$$X_{j,i_1}(t_{i_1}) \ge R_{j,i_1} w_{\varpi(1)}, \ \forall t_{i_1} \ge S^L_{1;i_1}. \tag{4.20}$$

Recall the definition of $\vec{R}_j$ in (4.1). Recall again (see (X1)(a) and (X2)(d)) the recursion

$$S^L_{1;i} = \rho_i Y_1, \ S^R_{k;i} = S^L_{k;i} + R_{i,i_k} w_{\varpi(k)}, \ k \in [N], \ \text{ and } S^L_{p;i} = S^R_{p-1;i}, \ p \in [N] \setminus \{1\}, \tag{4.21}$$

for each $i \in [m]$, where as usual $i_k = \mathbf{t}(\varpi(k))$. In particular, using (4.19), (4.20), and the middle identity in (4.21) we see that if $j \neq i_1$ then for any $\vec{t}$ such that $t_{i_1} \geq S^L_{1;i_1}$ and $t_j \leq S^R_{1;j}$

$$\text{we have } \; X_j(\vec{t}) = \sum_{i=1}^m X_{j,i}(t_i) \geq \underbrace{R_{j,i_1} w_{\varpi(1)}}_{X_{j,i_1}(t_{i_1})} - \underbrace{S^R_{1;j}}_{X_{j,j}(t_j)} = -\rho_j Y_1. \tag{4.22}$$

Similarly, if $j = i_1$ we apply (4.18), monontonicity of off-diagonal processes $X_{i_1,j}(\cdot)$, $j \neq i_1$, and again (4.21) to conclude that for any $\vec{t}$ such that$t_{i_1} \in [S^L_{1;i_1}, S^R_{1;i_1}]$

$$\text{we have } \; X_{i_1}(\vec{t}) = \sum_{i=1}^m X_{i_1,i}(t_i) \geq -S^R_{k;i_1} + w_{\varpi(1)} = -\rho_{i_1} Y_1. \tag{4.23}$$

The estimates in (4.22)–(4.23), combined with the fact that $\mathbf{T}(Y_1) =^{a.s} \vec{S}^L_1$ immediately imply that $\mathbf{T}(Y_1 + \epsilon) > \vec{S}^R_1$, almost surely, which is the base of our induction argument for (4.13).

Now suppose that we have shown $\{k-1 \leq N \leq K\} \subset \cap_{\varepsilon>0}\{\mathbf{T}(Y_1+\epsilon) > \vec{S}^R_{k-1}\}$. We wish to show that $\{k \leq N \leq K\} \subset \cap_{\varepsilon>0}\{\mathbf{T}(Y_1 + \epsilon) > \vec{S}^R_k\}$. Recalling $\vec{S}^R_{k-1} = \vec{S}^L_k$, we proceed (as in the above analysis on $[\vec{S}^L_1, \vec{S}^R_1]$) to estimate $\mathbb{X}$ from below on $[\vec{S}^L_k, \vec{S}^R_k]$. More precisely, observe that if $\vec{t} \geq \vec{S}^L_k = \vec{S}^R_{k-1}$ is such that $t_j \leq S^R_{k;j}$ for some $j \in [m]$ then

$$\begin{aligned}
\sum_{i=1}^m X_{j,i}(t_i) &\geq \underbrace{\sum_{i \neq j} \sum_{p \leq k : i_p = i} R_{j,i_p} w_{\varpi(p)}}_{(4.17)} + \underbrace{\sum_{p \leq k : i_p = j} w_{\varpi(p)} - S^R_{k;j}}_{(4.18)} \\
&= \sum_{p \leq k} R_{j,i_p} w_{\varpi(p)} - S^R_{k;j} \\
&= -S^L_{1;j} = -\rho_j Y_1.
\end{aligned}$$

In the second line above we combined the two summations and used $R_{j,j} = 1$. For the final two identities we again applied (4.21) and used telescoping. We obtain $\{k \leq N \leq K\} \subset \cap_{\varepsilon>0}\{\mathbf{T}(Y_1 + \epsilon) > \vec{S}^R_k\}$, which concludes the induction step and establishes (4.10).

Now consider

$$\tilde{\mathbb{X}}(\vec{t}) := \mathbb{X}(\vec{t} + \vec{S}^R_N) - \mathbb{X}(\vec{S}^R_N), \; \vec{t} \geq \vec{0}.$$

Stopping time $\vec{S}^R_N$ is the direct analogue of $\vec{0}$, while $\vec{S}^L_{N+1} - \vec{S}^R_N$ (if finite, that is well-defined) is the direct analogue of $\vec{S}^L_1$, where $\widetilde{\mathbb{X}}(\cdot)$ now takes the role of $\mathbb{X}(\cdot)$. Therefore, the whole argument above can be repeated for the exploration of the second, and iteratively, of the pth connected component of DCSBM intersecting $[m]_{\vec{\rho}}$. In addition, (4.9) and Lemma 4.11 apply directly on the shifted process $\widetilde{\mathbb{X}}$ (and its iterations), and imply in particular that

$$\widetilde{\mathbf{T}}(y) = y\vec{\rho}, \text{ iff } y \leq \widetilde{Y}_1,$$

where clearly on the event $\{N+1 \leq K\}$ we have $\widetilde{Y}_1 \equiv Y_2$. In addition, note that for any $y > 0$ (due to (4.14)) we have $\widetilde{\mathbf{T}}(y) := \inf\{\vec{t} \geq \vec{0} : \widetilde{\mathbb{X}}(\vec{t}-) \leq -y\vec{\rho}\} \equiv \inf\{\vec{t} \geq \vec{S}^R_N : \mathbb{X}(\vec{t}-) \leq -(y + Y_1)\vec{\rho}\} - \vec{S}^R_N =: \mathbf{T}(y + Y_1) - \vec{S}^R_N$, almost surely.

With these correspondencies, the above linear expression for $\widetilde{\mathbf{T}}$ becomes $\mathbf{T}(Y_1 + y) = \vec{S}^R_N + y\vec{\rho} = \mathbf{T}(Y_1+) + y\vec{\rho}$, where $y \in (0, Y_2]$, and this is clearly equivalent to the (affine map) expression in the statement of the lemma. $\square$

4.5 Proof of Theorem 2.6

In previous sections we did most of the work needed for completing the proof of Theorem 2.6.

The fact that the connected components appear in a size-biased order follows from the construction of Exploration 4.2 and the equivalence with Exploration 4.1 in Proposition 4.3. Indeed, let $\{\mathcal{C}_j; j \geq 1\}$ be the connected components of $\mathcal{G}$, and recall that $\mathscr{S}(\mathcal{C}_j) = \sum_{(l,i)\in\mathcal{C}_j} \rho_i Q_{i,i} w_l^i$. The size-biased ordering with respect to the weights $(\mathscr{S}(\mathcal{C}_j))_j$ is conventionally done as follows: let

$$E_l \,|\, \sigma(\mathcal{C}_j,\ j \geq 1) \sim \mathrm{Exp}(\mathscr{S}(\mathcal{C}_l)),\ \ l \geq 1, \tag{4.24}$$

be conditionally independent exponentials; the size-biased ordering $(\mathcal{C}_{\tau_k})_k$ is defined through a random permutation $(\tau_k)_k$, which is specified via

$$E_{\tau_k} < E_{\tau_{k+1}},\ \ k \geq 1,\ \text{almost surely}.$$

The random ordering of components obtained in Exploration 4.2 is different from the conventional ordering via (4.24) in at least two ways: (a) we use fragments of connected component weights (when searching for the next "root vertex") instead of full connected component weights, and (b) we draw (conditionally independent) exponential variables sequentially rather than simultaneously (this corresponds to our gradual exploration of $\mathcal{G}$).
Nevertheless, the two approaches can be easily linked due to these elementary properties: (i) the minimum of n independent exponetial random variables is again an exponential variable with rate equal to the sum of n individuals rates, and (ii) $(E_{\tau_k})_k = (\min_j E_j, \min_{j\neq\tau_1} E_j, \min_{j\notin\{\tau_1,\tau_2\}} E_j, \ldots)$.

Now suppose that we are given another family $(F_v)_{v\in\mathcal{G}}$ of (conditionally and) mutually independent exponentials given $(\mathcal{C}_j)_{j\geq 1}$, such that

$$F_v \,|\, \sigma(\mathcal{C}_j,\ j \geq 1) \sim \mathrm{Exp}(\rho_{\mathbf{t}(v)} Q_{\mathbf{t}(v),\mathbf{t}(v)} w_v),\ \ v \in \mathcal{G}.$$

Due to (i) above, we have

$$E_l \,|\, \sigma(\mathcal{C}_j,\ j \geq 1) \stackrel{d}{=} \min_{v\in\mathcal{C}_l} F_v |\, \sigma(\mathcal{C}_j,\ j \geq 1).$$

Let $\mathcal{C}(v)$ denote the connected component of $\mathcal{G}$ which contains v. If $V_1 := \mathrm{argmin}_{v\in\mathcal{G}} F_v$ let $V_2 := \mathrm{argmin}_{v\in\mathcal{G}\setminus\mathcal{C}(V_1)} F_v$, $V_3 := \mathrm{argmin}_{v\in\mathcal{G}\setminus(\mathcal{C}(V_1)\cup\mathcal{C}(V_2))} F_v$, and continue inductively. The reader should note that here we confound again the graph with its vertex set, for ease of notation.

The main point of this discussion is the observation that, due to (i) and (ii) above, we have

$$(E_{\tau_k})_k \stackrel{d}{=} (\min_v F_v, \min_{v\in\mathcal{G}\setminus\mathcal{C}(V_1)} F_v, \min_{v\in\mathcal{G}\setminus(\mathcal{C}(V_1)\cup\mathcal{C}(V_2))} F_v, \ldots). \tag{4.25}$$

We leave it to the interested reader to check that Exploration 4.2 in step (G1)(a) gradually produces a sequence of exponential random variables equally distributed as the right-hand-side in (4.25).

Let $\mathcal{G} = \mathcal{G}(\mathbf{W}, Q)$ and recall the notation given before the statement of Theorem 2.6. Given $\mathcal{G}$, let us generate conditionally independent exponential random variables $(E_r;\, r \geq 1)$ with respective rates $(\mathscr{S}(r);\, r \geq 1)$. Combining the formulae for $\mathbf{T}$ in Lemma 4.12 with the size-biased ordering of connected components yields the following.

Corollary 4.13. *We have*

$$(\mathbf{T}(y);\, y \geq 0) \stackrel{d}{=} \left(\vec{\rho} y + \sum_r R\vec{\mathscr{M}}(r)\, 1_{\{E_r<y\}};\, y \geq 0\right).$$

The strict inequality in the event of the rth indicator function is completely consistent with Lemma 4.12 and the left continuity of $\mathbf{T}$. While Lemma 4.12 does not concentrate on the behavior of $\mathbf{T}$ after its final (ζ_Kth) jump, it is easy to see that on $(\sum_{i=1}^{\zeta_K} Y_i, +\infty)$ $\mathbf{T}$ evolves as a deterministic affine map, parallel to the line $y \mapsto y\vec{\rho}$.

5 Excursion representation: preliminaries and special case

This section is devoted to the analysis of fields and their hitting times in the deterministic setting where the fields are well-behaved. In the next section, we will explore the general case, but this special case is instructive for our later construction and motivation. We start by establishing some notation which will be useful throughout the sequel.

5.1 Notation

We will define several classes of functions with the following inclusions

$$D_0^{\uparrow\uparrow}(\mathbb{R}_+) \subset D_0^{\uparrow}(\mathbb{R}_+) \subset D_0^{+}(\mathbb{R}_+) \subset D_0(\mathbb{R}_+) \subset D(\mathbb{R}_+).$$

As usual, we denote by $D([0,\infty)) = D(\mathbb{R}_+)$ the Skorohod space of real càdlàg functions on $\mathbb{R}_+$. Let $D_0(\mathbb{R}_+) \subset D(\mathbb{R}_+)$ contain all $f \in D(\mathbb{R}_+)$ such that $f(0) = 0$. Note that all $f \in D(\mathbb{R}_+)$ is necessarily continuous at 0, and moreover if $f \in D_0(\mathbb{R}_+)$ then $\lim_{t\downarrow 0} f(t) = 0$. Furthermore, $D_0^{+}(\mathbb{R}_+)$ denotes the collection of $f \in D_0(\mathbb{R}_+)$ with no negative jumps, or equivalently, such that $f(t) \geq f(t-)$ for all t. A strictly smaller class $D_0^{\uparrow}(\mathbb{R}_+)$ contains all the non-decreasing $f \in D_0(\mathbb{R}_+)$. Finally, $D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ denotes the collection of $f \in D_0^{\uparrow}(\mathbb{R}_+)$, such that $f(t) > 0$ for all $t > 0$ and $f(t) \to \infty$ as $t \to \infty$. Equivalently, $f \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ if and only if it is non-decreasing, strictly increasing from the right at 0, and unbounded on $\mathbb{R}_+$. Given an $f \in D(\mathbb{R}_+)$ we will write

$$\mathcal{J}(f) = \{t : f(t) \neq f(t-)\}$$

as the collection of jump times of a function f.

We will focus on "well-behaved" fields $\mathbb{x} \in D(\mathbb{R}_+^m)$ where

$$D(\mathbb{R}_+^m) := \left\{ \mathbb{x} = (x_1, \cdots, x_m) \;\middle|\; \begin{array}{c} x_i(\vec{t}) = \sum_{j=1}^{m} x_{i,j}(t_j), \\ x_{i,j} \in D_0^{\uparrow}(\mathbb{R}_+),\ \forall j \neq i \text{ and } x_{i,i} \in D_0^{+}(\mathbb{R}_+) \end{array}, \ \forall i \in [m] \right\}.$$

Assume that we are given some $\mathbb{x} = (x_i)_{i\in[m]} \in D(\mathbb{R}_+^m)$ and $\vec{\rho} \in \mathbb{R}_+^m$, $\vec{\rho} \neq \vec{0}$. Recalling (2.6), for each $y \geq 0$ we write $\mathbf{T}^{\vec{\rho}}(\mathbb{x}; y)$ or $\mathbf{T}(y)$ (when $\mathbb{x}, \vec{\rho}$ are specified by context) for $\mathbf{T}(\mathbb{x}; y\vec{\rho})$, which is the *minimal solution* of

$$x_i(\vec{t}-) = -\rho_i y, \quad \forall i \in [m] \text{ s.t. } t_i < \infty. \tag{5.1}$$

Abusing notation, we will shortly write (5.1) in the form

$$\mathbb{x}(\vec{t}-) = -\vec{\rho} y. \tag{5.2}$$

Note that $\mathbf{T}^{\vec{\rho}}(\mathbb{x}; y) \in [0,\infty]^m$ is well-defined, according to [16, Lemma 2.3]. Moreover, if $\mathbf{T}^{\vec{\rho}}(\mathbb{x}; y) \in \mathbb{R}_+^m$, then also $\mathbf{T}^{\vec{\rho}}(\mathbb{x}; u) \in \mathbb{R}_+^m$ for all $u \in [0, y)$, since in fact $\mathbf{T}^{\vec{\rho}}(\mathbb{x}; u) \leq \mathbf{T}^{\vec{\rho}}(\mathbb{x}; y)$.

We will frequently omit the symbols $\vec{\rho}$ and $\mathbb{x}$ from the notation if they are clear from context. We also write the coordinates

$$\mathbf{T}(y) = (T_1(y), T_2(y), \cdots, T_m(y)) \in [0, \infty]^m.$$

Since our goal is to prove Theorem 2.8, we also recall the additional assumption (2.15). This could be called the "column-wise off-diagonal proportionality" (for probabilistic interpretations see Section 2.5): there is some $\vec{\rho} \in (0, \infty)^m$ such that for each $l \in [m]$ and all $i, j \neq l$

$$\frac{x_{i,l}(t)}{\rho_i} = \frac{x_{j,l}(t)}{\rho_j}, \quad \text{for all } t \geq 0. \tag{5.3}$$

Under this additional hypothesis, we can and will introduce the following notation

$$x_{*,l}(t) := \frac{x_{j,l}(t)}{\rho_j}, \quad t \geq 0, \tag{5.4}$$

where j is any element of $[m] \setminus l$. Let

$$D_{\vec{\rho}}(\mathbb{R}_+^m) \subset D(\mathbb{R}_+^m)$$

denote the collection of all fields $\mathbb{x} \in D(\mathbb{R}_+^m)$ which also satisfy (5.3).

If f is a real function of a real variable, let us denote by $\underline{f}$ the "past infimum" of f:

$$\underline{f}(t) = \inf_{r \in [0,t]} f(r), \quad t \geq 0.$$

In the sequel we will often work with $\underline{x}_{i,i}$ instead of $x_{i,i}$ for $i \in [m]$. The main advantage of $\underline{x}_{i,i}$ over $x_{i,i}$ is its continuity (here we use the fact that $x_{i,i} \in D_0^+(\mathbb{R}_+)$) and monotonicity. The following analogue of x_i

$$\underline{x}_i(\vec{t}) := \underline{x}_{i,i}(t_i) + \sum_{j \neq i} x_{i,j}(t_j), \quad \vec{t} \in \mathbb{R}_+^m, \tag{5.5}$$

will be particularly useful to us. We naturally write $\underline{\mathbb{x}}$ for $(\underline{x}_1, \ldots, \underline{x}_m)$.

We will be interested in curves $\vec{\gamma} : \mathbb{R}_+ \to \mathbb{R}_+^m$ such that γ_i is non-decreasing for each $i \in [m]$. For $\vec{l}, \vec{r} \in \mathbb{R}^m$ we recall that $\vec{l} < \vec{r}$ (resp. $\vec{l} \leq \vec{r}$) if $l_i < r_i$ (resp. $l_i \leq r_i$) for each $i \in [m]$. We also set $\vec{r} \pm \delta = (r_i \pm \delta)_{i \in [m]}$ for $\vec{r} \in \mathbb{R}^m$ and any $\delta > 0$ a scalar. Let $(\vec{l}, \vec{r}) = \prod_{i=1}^m (l_i, r_i)$ for $\vec{l} < \vec{r}$.

5.2 Preliminary properties of $\mathbf{T}(y)$

Recall (5.1) and (5.2). In particular, $\mathbf{T}(y) \equiv \mathbf{T}^{\vec{\rho}}(\mathbb{x}; y) = (T_1(y), \ldots, T_m(y))$ is the (component-wise) minimizer of (5.2). Hence if $\mathrm{sol}(y) := \{\vec{r} = (r_1, r_2, \ldots, r_m) \geq \vec{0} : \vec{r} \text{ solves } (5.2)\}$ then

$$T_i(y) = \min_{\vec{r} \in \mathrm{sol}(y)} r_i, \quad \forall i \in [m]. \tag{5.6}$$

It is easy to see that the original minimization problem (with equality) is solution equivalent to the one (with inequalities) where the component-wise minimum is chosen from

$$\{\vec{r} = (r_1, r_2, \ldots, r_m) \geq \vec{0} : \mathbb{x}(\vec{r}-) \leq -y\vec{\rho}\} \tag{5.7}$$

instead of $\mathrm{sol}(y)$. Furthermore, let us denote by $\|\vec{t}\|_1 = \sum_i t_i$ the usual ℓ^1-norm of $\vec{t}$. In the setting where all the components of $\mathbf{T}(y)$ are finite, we have the following useful equivalence.

Lemma 5.1. *Provided that $\mathbf{T}(y) \in \mathbb{R}^m_+$, the above optimization (minimization) problem for $\mathbf{T}(y)$ is equivalent to*

$$\begin{cases} \mathbb{x}(\vec{t}-) = -\vec{\rho}y, \\ \|\vec{t}\|_1 \to \min . \end{cases} \tag{5.8}$$

Proof. Due to $\mathrm{sol}(y) \subset [0,\infty)^m$ and (5.6), the component-wise minimum over $\mathrm{sol}(y)$ is also the vector in $\mathrm{sol}(y)$ which minimizes the l_1-norm. □

It is straight-forward from (5.2) that

$$\frac{x_i(\mathbf{T}(y)-)}{\rho_i} = -y, \quad i \in [m],$$

if $\mathbf{T}(y) \in \mathbb{R}^m_+$. Moreover, recalling (5.5), we get the following.

Lemma 5.2. *Let $\mathbf{T}(y)$, $y \geq 0$, be as defined above and suppose $\mathbf{T}(y) \in \mathbb{R}^m_+$. Then, for each $y \geq 0$, $\mathbf{T}(y)$ is*

i) the component-wise minimizer of

$$\underline{\mathbb{x}}(\vec{t}-) = -\vec{\rho}y, \tag{5.9}$$

and the solution to

$$\begin{cases} \underline{\mathbb{x}}(\vec{t}-) = -\vec{\rho}y, \\ \|\vec{t}\|_1 \to \min, \end{cases} \tag{5.10}$$

ii) $\underline{x}_{i,i}(t) > \underline{x}_{i,i}(T_i(y))$, for each $i \in [m]$ and every $t < T_i(y)$.

Proof. The first statement is a simple consequence of the definitions and the hypotheses. It also follows from [16, Lemma 2.3.4].

In order to show ii), suppose that for some $t^* < T_i(y)$ we have $\underline{x}_{i,i}(t^*) \leq \underline{x}_{i,i}(T_i(y))$. Then it is easy to see that the vector $\vec{t^*} := (T_1(y), \ldots, T_{i-1}(y), t^*, T_{i+1}(y), \ldots, T_m(y))$, which is strictly smaller than $\mathbf{T}(y)$ in component i, satisfies

$$\underline{\mathbb{x}}(\vec{t^*}-) \leq \underline{\mathbb{x}}(\mathbf{T}(y)-) = -\vec{\rho}y,$$

contradicting the minimality of $\mathbf{T}(y)$ (in the minimizing problem (5.7)). □

Lemma 5.3. *The map $y \mapsto T_i(y)$ is strictly increasing and left-continuous for each $i \in [m]$.*

Proof. As commented already, it is clear from the properties of $\mathbb{x}$ and $\underline{\mathbb{x}}$ (inherited from those of $x_{i,j}$ as i, j range through $[m]$) that $y \mapsto T_i(y)$ is non-decreasing for each i.

Suppose that $y' > y$, so that $-\vec{\rho}y' < -\vec{\rho}y$. Assuming there would be at least one $i \in [m]$ such that $T_i(y) = T_i(y')$, we could expand (using the non-strict monotonicity of T_j for each $j \in [m]$, together with the monotonicity of $x_{i,j}$ for all $i \neq j$)

$$\begin{aligned} -\rho_i y = \underline{x}_i(\mathbf{T}(y)-) &= \underline{x}_{i,i}(T_i(y)) + \sum_{j \neq i} x_{i,j}(T_j(y)-) \\ &= \underline{x}_{i,i}(T_i(y')) + \sum_{j \neq i} x_{i,j}(T_j(y)-) \\ &\leq \underline{x}_{i,i}(T_i(y')) + \sum_{j \neq i} x_{i,j}(T_j(y')-) = \underline{x}_i(\mathbf{T}(y')-) = -\rho_i y', \end{aligned}$$

which leads to a contradiction.

Note that [16, Lemma 2.3 4.] implies the left-continuity of $\mathbf{T}$. This can be verified directly by taking a sequence $y_n \nearrow y$, defining $\mathbf{T}^*(y) := \lim_n \mathbf{T}(y_n) = \sup_n \mathbf{T}(y_n)$ and using the monotonicity of $y \mapsto \mathbf{T}(y)$ to get $\mathbf{T}^*(y) \leq \mathbf{T}(y)$, and the minimality of $\mathbf{T}(y)$ to get the reversed inequality. □

Remark 5.4. It is not surprising that $\mathbf{T}$ is left-continuous, since it is an m-dimensional analogue of the left-continuous generalized inverse (the inequality in (5.7) is not strict). Since $\mathbf{T}$ is also (component-wise) strictly increasing, it is easy to see that it is in particular a non-decreasing left-continuous map with right limits (*ndlcrl* for short) function.

Lemma 5.2 is quite helpful when solving for $\mathbf{T}(y)$. Indeed, since $\mathbf{T}(y)$ is the solution of (5.10) we now also know that

$$\frac{\underline{x}_i(\mathbf{T}(y)-)}{\rho_i} = -y, \quad i \in [m],$$

and, therefore, that $\mathbf{T}(y)$ solves

$$\begin{cases} \dfrac{\underline{x}_1(\vec{t}-)}{\rho_1} = \dfrac{\underline{x}_i(\vec{t}-)}{\rho_i} = -y, \quad \forall i \in [m], \\ \\ \|\vec{t}\|_1 \to \min . \end{cases} \tag{5.11}$$

5.3 Solving for $\mathbf{T}(y)$ – Special Case

Recall (5.4) and the definition of $D_{\vec{\rho}}(\mathbb{R}^m_+)$. In this section and in the next section we assume that $\mathbb{x} \in D_{\vec{\rho}}(\mathbb{R}^m_+)$ for some given $\vec{\rho}$. Our aim is to construct a continuous curve $\vec{\gamma} : [0, \infty) \to \mathbb{R}^m$ whose values will contain $(\mathbf{T}^{\vec{\rho}}(\mathbb{x}; y),\ y \geq 0)$. This will enable us to encode the jumps of $\mathbf{T}^{\vec{\rho}} = \mathbf{T}$ by considering the excursions above past minima of real-valued functions $x_i(\vec{\gamma}(s))$, $s \geq 0$, $i \in [m]$. We will arrive to a suitable choice of $\vec{\gamma}$ by trying to solve for $\mathbf{T}(y)$.

For each $i \in [m]$ let us define

$$g_i(t) := x_{*,i}(t) - \frac{\underline{x}_{i,i}(t)}{\rho_i}, \quad t \geq 0. \tag{5.12}$$

It is easy to check from the definitions (of $D_{\vec{\rho}}(\mathbb{R}^m_+)$ in particular) that $g_i \in D_0^{\uparrow}(\mathbb{R}_+)$ for each $i \in [m]$.

After elementary algebraic manipulations (including several cancellations due to (2.15)) the optimization problem (5.11) can be replaced by

$$\begin{cases} x_1(\vec{t}-) = -\rho_1 y, \\ g_1(t_1-) = g_i(t_i-), \quad \forall i \in [m], \\ \|\vec{t}\|_1 \to \min . \end{cases} \tag{5.13}$$

Indeed, the constraint $\underline{x}_1(\vec{t}-) = -\rho_1 y$ can be replaced by $x_1(\vec{t}-) = -\rho_1 y$ due to [16, Lemma 2.3.4], or alternatively due to the fact that the component-wise minimal solution of

$$x_1(\vec{t}-) = -\rho_1 y, \ \ \underline{x}_i(\vec{t}-) = -\rho_i z, \ \forall i \in [m], \text{ and some } z \geq 0,$$

is again necessarily equal to $\mathbf{T}(y)$ (the parameter z is free, and we can use monotonicity of $z \mapsto \mathbf{T}(z)$). At this intermediate step we know that $\mathbf{T}(y)$ is uniquely determined as the solution of (5.13).

In order to circumvent several technical issues, we presently make the following additional assumption (this hypothesis is dropped in Section 6.2 due to a novel concept and a considerable amount of additional work):

$$x_{i,j}(t) \text{ are strictly increasing and continuous for all } i \neq j. \tag{SC}$$

In particular, for each i, $t \mapsto g_i(t)$ is a strictly increasing continuous function such that $g_i(0) = 0$. We also suppose that $g_i(t) \to \infty$ as $t \to \infty$ for all i, which can also by stated by a more complicated assumption placed on the field $\mathbb{x}$. Consequently, each g_i is a homeomorphism, and we let g_i^{-1} denote its inverse.

Remark 5.5. Anticipating analysis in Section 6.2, it is practical for us to continue writing $\vec{t}-$ instead of $\vec{t}$ in constraints involving $\mathbb{x}$.

The above strict monotonicity and continuity implies that for any given y

$$g_i(T_i(y)-) = g_i(T_i(y)), \text{ or equivalently, that } x_{*,i}(T_i(y)-) = x_{*,i}(T_i(y)), \quad \forall i \in [m], \tag{A1}$$

and also that

$$g_i^{-1} \circ g_i(T_i(y)) = T_i(y), \quad \forall i \in [m]. \tag{A2}$$

Since $\mathbf{T}(y)$ solves $g_1(t_1-) = g_i(t_i-)$ according to (5.13), the two properties (A1)–(A2) yield

$$T_i(y) = g_i^{-1} \circ g_1(T_1(y)). \tag{5.14}$$

In particular, we see that

$$\mathbf{T}(y) = (T_1(y), g_2^{-1} \circ g_1(T_1(y)), \dots, g_m^{-1} \circ g_1(T_1(y))),$$

where $T_1(y)$ is the minimizer of

$$\begin{cases} x_1(t_1-, g_2^{-1} \circ g_1(t_1)-, \dots, g_m^{-1} \circ g_1(t_1)-) = -\rho_1 y, \\ t_1 + \sum_{i \neq 1} g_i^{-1} \circ g_1(t_1) \to \min. \end{cases}$$

This analysis can be improved by introducing a convenient reparametrization of (5.13):

$$\begin{cases} x_1(\vec{t}-) = -\rho_1 y, \\ g_1(t_1-) = g_i(t_i-), \quad \forall i \in [m], \\ s = \sum_{i=1}^m t_i, \\ s \to \min. \end{cases} \tag{5.15}$$

Define

$$f(u) := \sum_{i=1}^m g_i^{-1}(u), \ u \geq 0, \ \text{ and } \ \kappa := f^{-1}, \tag{5.16}$$

where κ is the usual inverse of the homeomorphism f.

Define

$$s(y) := \sum_{i=1}^m T_i(y) = \|\mathbf{T}(y)\|_1, \tag{5.17}$$

and note that $y \mapsto s(y)$ is again left-continuous (in fact it is a ndlcrl map), admitting at most countably many points of discontinuity. Recalling (5.14), we see that if $u = g_1(T_1(y)-)$ then

$g_i^{-1}(u) = T_i(y)$ and so $f(u) = s(y)$, at least in the case when $\mathbf{T}(y) \in \mathbb{R}_+^m$. Hence we deduce that $(\mathbf{T}(y), s(y))$ solves

$$\begin{cases} x_1(\vec{t}-) = -\rho_1 y, \\ t_i = g_i^{-1} \circ g_1(t_1), \quad \forall i \in [m], \\ s = f(g_1(t_1)), \\ s \to \min, \end{cases} \tag{5.18}$$

Note that f is strictly increasing, since the auxilliary functions $(g_i)_{i\in[m]}$ are assumed to be strictly increasing and continuous. Using (A1) and the reasoning of the previous paragraph we conclude that $f(g_1(T_1(y)-)) = f(g_1(T_1(y))) = s(y)$. Applying the inverse $\kappa = f^{-1}$ to the last equation we see that $g_1(T_1(y)) = \kappa(s(y))$. It is now immediate from (A2) or from (5.18) that

$$T_1(y) = g_1^{-1} \circ \kappa(s(y)) = g_1^{-1} \circ \kappa(\|T(y)\|_1),$$

where $s(y)$ is the minimal solution of (5.18). Note that there is nothing special with $i = 1$, which we initially took in (5.13) as the reference index. The same reasoning as above leads to the following conclusion: under assumptions (A1)–(A2) we get

$$\mathbf{T}(y) = (\vec{\gamma}_{sc}(s(y))), \tag{5.19}$$

where

$$\vec{\gamma}_{sc}(s) := (g_1^{-1} \circ \kappa(s), \ldots, g_m^{-1} \circ \kappa(s)),$$

and where $s(y)$ is the minimal solution to

$$x_j(\vec{\gamma}_{sc}(s)-) = -\rho_j y, \tag{5.20}$$

for any (and every) $j \in [m]$. Here the subscript "sc" just stands for special case.

6 Excursion representation: smooth composition and the general case

Assumptions (A1)–(A2) are cumbersome to check and false in general. Indeed, (SC) is almost surely false for the fields constructed in (2.4). We will soon construct a generalization of the "composed with" operator which proves quite useful in the sequel, and might be of general interest. As far as we are aware, this novel concept, which could be studied on the level of undergraduate calculus, has not yet appear in the literature.

6.1 Smooth compositions

Recall briefly the technical issues of our construction of $\vec{\gamma}_{sc}$ in the previous section. More precisely, these are the steps in the previous construction of the solution $\mathbf{T}(y)$ which would fail (in the sense of mathematical rigor) without (SC) or (A1)–(A2).

In this section, we no longer assume (SC). The auxiliary functions $(g_i)_{i\in[m]}$ are no longer homeomorphisms, however they are still non-decreasing and right-continuous functions. In the next section we will assume that $g_i \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ for each $i \in [m]$. We henceforth write *ndrcll* for non-decreasing right-continuous functions with left-limits. From now on g_i^{-1} will denote the *generalized inverse* of g_i. Recall that if h is ndrcll, its (right-continuous generalized) inverse h^{-1} is defined as follows

$$h^{-1}(s) := \inf\{u > 0 \,:\, h(u) > s\}. \tag{6.1}$$

The construction of $\vec{\gamma}_{sc}$ relied on properties (A1) and (A2). The former requires some smoothness, and the latter is trickier to mimic since $h^{-1} \circ h$ need not equal the identity. Our composition-like operator $\tilde{\circ}$ constructed below is such that

$$g \,\tilde{\circ}\, g^{-1} = g^{-1} \,\tilde{\circ}\, g = \mathrm{id} \tag{6.2}$$

for each $g \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$.

Let us recall a few basic useful inequalities, which relate a ndrcll map and its inverse:

$$h(u) > s \quad \Longrightarrow \quad h^{-1}(s) \leq u, \tag{6.3}$$

$$h(u) \leq s \quad \Longrightarrow \quad h^{-1}(s) \geq u, \tag{6.4}$$

and by contraposition of (6.3)

$$h^{-1}(s) > u \quad \Longrightarrow \quad h(u) \leq s. \tag{6.5}$$

It is also easy to see that

$$(h^{-1})^{-1} = h.$$

One can see [37, Chapter 0] for more information.

Before turning to the construction of $\tilde{\circ}$, let us recall some standard properties of the inverse, most of which will be used without further mention in the sequel. These are all elementary consequences of definitions and inequalities (6.3)–(6.5), and their proofs are left to the reader. We recall that $\mathcal{J}(f)$ is the collection of jump times for a function f.

Lemma 6.1. *Let $h \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ and h^{-1} be as in (6.1).*

i) If h is strictly increasing from the right at some $u \geq 0$, then $h^{-1}(h(u)) = u$.

ii) If $h(h^{-1}(s)) = s$ for some $s \geq 0$, then h^{-1} is strictly increasing from the right at s.

iii) $h^{-1}(h(u-)) \geq u$ and $h^{-1}(h(u)-) \leq u$ for all $u \geq 0$. In particular, if $h(u-) < h(u)$ (i.e. $u \in \mathcal{J}(h)$) for some $u \geq 0$, then $h^{-1}(h(u-)) = h^{-1}(h(u)-) = u$.

vi) If $h(h^{-1}(s)-) = h(h^{-1}(s))$ for some $s \geq 0$, then $h(h^{-1}(s)) = s$.

Remark 6.2. Clearly for $g \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$, $g(u) > g(u-)$ (i.e. $u \in \mathcal{J}(g)$) if and only if the inverse image $(g^{-1})^{-1}(\{u\})$ of g^{-1} at u is the (positive length) interval $[g(u-), g(u)]$, provided g is strictly increasing from the right at u, or $[g(u-), g(u))$ otherwise. Note that $g \circ g^{-1} = \mathrm{id}$ except on the union of $(g^{-1})^{-1}(\{u\})$ (in some cases, taken without the right boundary point), over all $u \in \mathcal{J}(g)$. The set of jump points for g is at most countable. However the above set of exceptions can be quite large. Indeed, for "pure jump" functions $g \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ we have that $\bigcup_{u \in \mathcal{J}(g)} (g^{-1})^{-1}(\{u\}) = (0, \infty)$. It will be convenient to use below an example of such a function

$$g_e(u) := \sum_{k=1}^{\infty} \frac{1}{k} \, 1_{[1/(k+1), 1/k)} + \sum_{j=1}^{\infty} j \, 1_{[j, j+1)}. \tag{6.6}$$

Let $D_{0,J}^{\uparrow}(\mathbb{R}_+)$ consist of all $g \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ such that g is both strictly increasing from the left and from the right at any jump point u of g. If $g \in D_{0,J}^{\uparrow}(\mathbb{R}_+)$ has a jump at $u > 0$ then

$(g^{-1})^{-1}(\{u\}) = [g(u-), g(u)]$ and moreover $g \circ g^{-1}(s) = s$ for $s = g(u)$. Therefore it is natural to define $g \,\tilde{\circ}\, g^{-1} \equiv g \circ g^{-1} = \mathrm{id}$ on the good set

$$G(g) := \left(\bigcup_u [g(u-), g(u)) \right)^c ,$$

and to define $g \,\tilde{\circ}\, g^{-1}$ as a *linear spline* on $G(g)^c$.

In particular $g \,\tilde{\circ}\, g^{-1}$ is defined on $[g(u-), g(u))$ by linearly interpolating through $(g(u-), \lim_{\{v \nearrow g(u-): v \in G(g)\}} g \,\tilde{\circ}\, g^{-1}(v)) = (g(u-), g(u-))$ and $(g(u), g \,\tilde{\circ}\, g^{-1}(g(u))) = (g(u), g(u))$. As a result we get $g \,\tilde{\circ}\, g^{-1} = \mathrm{id}$.

Remark 6.3. (a) The good set $G(g)$ is dense in any left neighborhood of $g(u-)$, using the fact that g jumps at u and the assumption $g \in D^{\uparrow}_{0,J}(\mathbb{R}_+)$. Therefore, it is possible to take the left limit above through points in $G(g)$.

(b) We could have just set $g \,\tilde{\circ}\, g^{-1} := \mathrm{id}$ instead of going through the above "construction by linear approximation", but the point here is that (under mild and natural "compatibility" assumptions, see Definition 6.4 below) the final step in the above construction can be repeated in the context where g is an element of $D^{\uparrow\uparrow}_0(\mathbb{R}_+)$, and where g^{-1} is replaced by another function $\kappa \in D^{\uparrow\uparrow}_0(\mathbb{R}_+)$.

Function g_e defined in (6.6) is an element of $D^{\uparrow\uparrow}_0(\mathbb{R}_+) \setminus D^{\uparrow}_{0,J}(\mathbb{R}_+)$ and, moreover, it is not strictly increasing from the right at any of its (countably many) jumps. Furthermore, the good set $G(g_e)$ consists of a single point 0. Consider any three of its consecutive jumps, these could be for example $u_i = i$, with $i = 1, 2, 3$ (the conclusion is the same in general). If $s_i := g_e(u_i)$, then $g_e \circ g_e^{-1}(s_i) = g_e(u_{i+1}) = s_{i+1}$, for $i = 1, 2$. One could be tempted to define of $g_e \,\tilde{\circ}\, g_e^{-1}$ by linearly interpolating through the points $(s_i, g_e \circ g_e^{-1}(s_i))$ and $(s_{i+1}, g_e \circ g_e^{-1}(s_{i+1}))$ on each $[s_i, s_{i+1}]$. The result is again a continuous function, but clearly different from the identity map.

However, one could extend $\tilde{\circ}$ in a different and better way.

Definition 6.4. We say that two elements g and κ of $D^{\uparrow\uparrow}_0(\mathbb{R}_+)$ are *compatible* if

$$\kappa^{-1}(\{u\}) = \{r \geq 0 : \ \kappa(r) = u\} \text{ has positive length whenever } u \in \mathcal{J}(g), \tag{H1}$$

$$\text{and } g(\kappa(s-)) = g(\kappa(s)-) \text{ whenever } \kappa(s-) < \kappa(s). \tag{H2}$$

Assumption (H1) means precisely that the inverse κ^{-1} of κ also jumps at each jump point u of g. In this case, the segment $\kappa^{-1}(\{u\})$ equals $[\kappa^{-1}(u-), \kappa^{-1}(u)]$ (resp. $[\kappa^{-1}(u-), \kappa^{-1}(u))$) provided κ^{-1} is strictly increasing from the right at u (resp. constant on $[u, u+\delta)$ for some $\delta > 0$).

We proceed by analogy to the construction of $g \,\tilde{\circ}\, g^{-1}$ for $g \in D^{\uparrow}_{0,J}(\mathbb{R}_+)$.

Definition 6.5. Suppose that the pair of functions (g, κ) satisfies (H1) and (H2). Define a new function $g \,\tilde{\circ}\, \kappa$ by

(i) if $s \notin \bigcup_{u \in \mathcal{J}(g)} [\kappa^{-1}(u-), \kappa^{-1}(u)]$ let $g \,\tilde{\circ}\, \kappa(s) = g \circ \kappa(s)$,

(ii) for each jump point $u \in \mathcal{J}(g)$, on $[\kappa^{-1}(u-), \kappa^{-1}(u)]$ define $g \,\tilde{\circ}\, \kappa$ to be the line segment through the points $(\kappa^{-1}(u-), g(u-))$ and $(\kappa^{-1}(u), g(u))$.

From now on we refer to $\tilde{\circ}$ as "smoothly composed with". As discussed immediately after Remark 6.3, there are situations where "smoothly composed with" and "composed with" differ on $(0, \infty)$, so $\tilde{\circ}$ is not a direct extension/generalization of $\circ$. If $\kappa = g^{-1}$ then it is easy to see that both (H1) and (H2) are satisfied and moreover that $g \,\tilde{\circ}\, \kappa = g \,\tilde{\circ}\, g^{-1} = \mathrm{id}$.

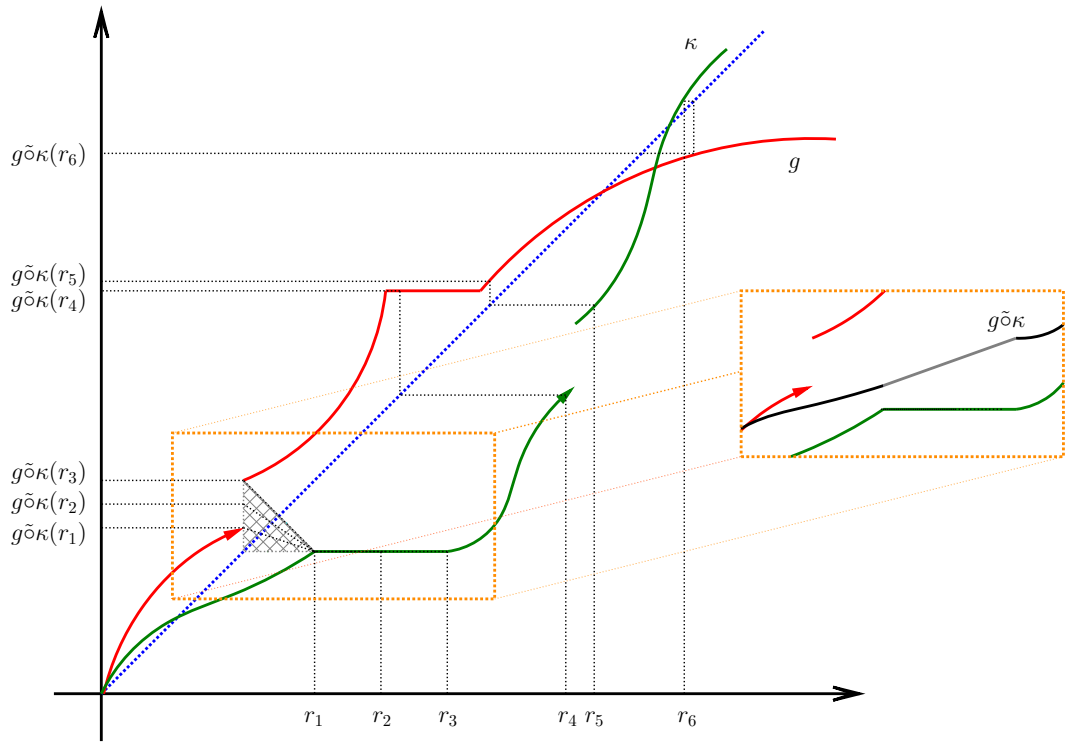

Figure 1: The red curve is the graph of g, while the green curve is the graph of κ. The rectangle with dotted orange borders is an interesting region, since it includes a jump u of g, as well as the image of κ on an interval which includes $[r_1, r_3] = [\kappa^{-1}(u-), \kappa^{-1}(u)]$. Its enlarged copy on the right depicts the graph of $\gamma = g \,\tilde{\circ}\, \kappa$ in blue. In particular, γ is a continuous (linear interpolation) function on $[r_1, r_3]$. Also note that (since g and κ are compatible) g must be constant on $[\kappa(s-), \kappa(s)]$, where s is the jump of κ in $[r_4, r_5]$.

6.1.1 Graphs of $g \circ \kappa$ and $g\tilde{\circ}\kappa$

Suppose that (g, κ) is a pair of compatible functions in $D_0^{\uparrow\uparrow}(\mathbb{R}_+)$. Let us assume that the graphs of g and κ are drawn in the same Cartesian system. (see Figure 1 for an illustration). Recall that in order to construct $g \circ \kappa(t)$ using one starts with the point $(t, 0)$ on the abscissa, searches for $(t, \kappa(t))$ on the graph of κ, from there moves horizontally to $(\kappa(t), \kappa(t))$ on the diagonal $y = x$; and finally searches along the vertical line $x = \kappa(t)$ for the point $(\kappa(t), g(\kappa(t))$ on the graph of g. This was done in the constructions of $g \,\tilde{\circ}\, \kappa(r_k) = g \circ \kappa(r_k)$ for $k \in \{4, 5, 6\}$ in Figure 1 (at these points κ is strictly increasing and continuous).
One can analogously construct $g \,\tilde{\circ}\, \kappa(r_j)$ for $j \in \{1, 2, 3\}$, knowing that $[\kappa^{-1}(u-), \kappa^{-1}(u)] = [r_1, r_3]$ for some jump point $u = \kappa(r_j)$ of g. In this general setting the final vertical move from $(u, u)(= (\kappa(r_j), \kappa(r_j))$ depends on j, or more precisely on the position of r_j within $[r_1, r_3]$. Since $r_1 = \min\{s : \kappa(s) = u\}$, the procedure ends at $(\kappa(r_1), g(u-))$, the "lowest point" with abscissa u in the closure of the graph of g. Since $r_3 = \max\{s : \kappa(s) = u\}$, the procedure ends at $(\kappa(r_3), g(u))$, the "highest point" with abscissa u in the closure of the graph of g. In general, any $r_2 \in [r_1, r_3]$ has representation $r_2(\lambda) = r_1 + \lambda(r_3 - r_1)$ for some $\lambda \in [0, 1]$. Given such λ, the procedure ends at $(\kappa(r_2), g(u-) + \lambda(g(u) - g(u-))$.

Remark 6.6. Our proofs below do not rely on an explicit formula for $g \,\tilde{\circ}\, \kappa$; however, we include it here for readers' benefit. Note that the family of sets $\{\kappa^{-1}(\{u\}),\ u \geq 0\}$ is a partition of $\mathbb{R}_+$.

Then for every $s, u \in \mathbb{R}_+$, such that $s \in \kappa^{-1}(\{u\})$

$$\begin{aligned}(g \,\tilde{\circ}\, \kappa(s)) &= g(u-) + \frac{g(u) - g(u-)}{\kappa^{-1}(u) - \kappa^{-1}(u-)}(s - \kappa^{-1}(u-)) \\ &= g(u) - \frac{g(u) - g(u-)}{\kappa^{-1}(u) - \kappa^{-1}(u-)}(\kappa^{-1}(u) - s).\end{aligned}$$

In the expression above, we assume that $A := \frac{g(u)-g(u-)}{\kappa^{-1}(u)-\kappa^{-1}(u-)}$ equals zero if g has no jump at u. Due to (H1) $\kappa^{-1}(u) - \kappa^{-1}(u-) > 0$ if $u \in \mathcal{J}(g)$, so A is always well defined.

Note in addition that it could be $\kappa^{-1}(v) - \kappa^{-1}(v-) > 0$ even if $g(v) = g(v-)$, and then the restriction of $g\tilde{\circ}\kappa$ on $[\kappa^{-1}(v-), \kappa^{-1}(v))$ is again a linear (constant) function $y(s) \equiv g \circ \kappa(v)$.

Even though the above remark is not necessary for the following lemma, using it makes the following straightforward:

Lemma 6.7. *Suppose that (g, κ) is a compatible pair. Then $g(\kappa(t)-) \leq g\tilde{\circ}\kappa(t) \leq g(\kappa(t))$ for all $t \geq 0$.*

Remark 6.8. (a) Note that each line segment in (ii) of Definition 6.5 is increasing (its slope, denoted by A in Remark 6.6, is strictly positive).

(b) If $v < u$ is another jump of g (and therefore of κ^{-1}) then $g(v-) < g(v) \leq g(u-) < g(u)$. We conclude that $g \,\tilde{\circ}\, \kappa$ restricted to $\bigcup_{u:\mathcal{J}(g)}[\kappa^{-1}(u-), \kappa^{-1}(u)]$ is a (strictly) increasing function.

(c) From (H1) and the right-continuity of κ at $\kappa^{-1}(u-)$ we conclude that at each jump point u of g it must be $\kappa(\kappa^{-1}(u-)) = u$, and so $g \circ \kappa(\kappa^{-1}(u-)) = g(u)$.

(d) The definition above does not rely on any additional properties of κ. In particular, it could be that $\kappa(\kappa^{-1}(u)) > u$, and also that $g(\kappa(\kappa^{-1}(u))) > g(u)$. Similarly, it may not be possible to access $\kappa^{-1}(u-)$ from the left through the good set of points in Definition 6.5 (i).

Remark 6.9. Condition (H1) is necessary for the interpolating line in Definition 6.5 (ii) to be well-defined. Also note that it could happen that for two successive jumps of g, occurring respectively at u_1 and $u_2 > u_1$, we have $\kappa^{-1}(u_1) = \kappa^{-1}(u_2-)$. This means that κ^{-1} is a constant function on $[u_1, u_2)$, and also that κ jumps at point $\kappa^{-1}(u_1) = \kappa^{-1}(u_2-)$ from value $u_1 = \kappa(\kappa^{-1}(u_1)-) = \kappa(\kappa^{-1}(u_2-)-)$ to value $u_2 = \kappa(\kappa^{-1}(u_1)) = \kappa(\kappa^{-1}(u_2-))$. Assumption (H2) with $s = \kappa^{-1}(u_1)$ is necessary for compatibility of the two different definitions of $g \,\tilde{\circ}\, \kappa$ at the point $\kappa^{-1}(u_1) = \kappa^{-1}(u_2-)$.

Lemma 6.10. *Let g, κ be elements of $D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ such that (g, κ) satisfies (H1) and (H2). Then, provided that $u \in \mathcal{J}(g)$ is a jump point of g,*
(a) $g \,\tilde{\circ}\, \kappa(\kappa^{-1}(u-)) = g(u-) \geq g \,\tilde{\circ}\, \kappa(s)$, for all $s < \kappa^{-1}(u-)$,
(b) $g \,\tilde{\circ}\, \kappa(\kappa^{-1}(u)) = g(u) \leq g \,\tilde{\circ}\, \kappa(s)$, for all $s > \kappa^{-1}(u)$.

In the argument below we will use several times and without explicit mention the fact that g (and κ) is ndrcll.

Proof. (a) If $s < \kappa^{-1}(u-)$ then necessarily $\kappa(s) < u$ so that $g \circ \kappa(s) \leq g(u-)$, and this implies the claim provided s is a "good point" from Definition 6.5 (i). Otherwise, it must be $s \in [\kappa^{-1}(v-), \kappa^{-1}(v)]$ for some $v < u$ and then $g \,\tilde{\circ}\, \kappa(s) \leq g(v) \leq g(u-)$.
(b) If $s > \kappa^{-1}(u)$, then necessarily $\kappa(s) > u$ so that $g \circ \kappa(s) \geq g(u)$, and this again implies the claim provided s is a "good point". Otherwise, it must be $s \in [\kappa^{-1}(v-), \kappa^{-1}(v)]$ for some $v > u$ and then $g \,\tilde{\circ}\, \kappa(s) \geq g(v-) \geq g(u)$. □

The above partial monotonicity result can be easily improved as follows.

Lemma 6.11. *Let g, κ be as in Lemma 6.10. Then $\gamma = g \tilde{\circ} \kappa$ is ndrcll.*

Proof. Since g and κ are elements of $D_0^{\uparrow\uparrow}(\mathbb{R}_+)$, the same is true for $g \circ \kappa$. To verify that $\gamma = g \tilde{\circ} \kappa$ is monotone non-decreasing, we use Lemmas 6.7, 6.10 when comparing the values of $\gamma(s_1)$ and $\gamma(s_2)$ if at least one, s_1 or s_2, is not a "good point" from Definition 6.5 (i), and we use the monotonoicity of $g \circ \kappa$ when both s_1 and s_2 are "good points". One can also try to derive this using Remark 6.6 and a comparison argument. Monotonicity implies that γ has both limits from the right and from the left at every point.

We will next show the right continuity of γ. By the construction of γ, it is trivial that γ is right-continuous on every interval $[\kappa^{-1}(u-), \kappa^{-1}(u))$, $u \geq 0$ (see also Remark 6.6). It remains to show that γ is right continuous on $A := \left(\bigcup_{u \geq 0} [\kappa^{-1}(u-), \kappa^{-1}(u))\right)^c$.

Let us fix $s \in A$. Then there exists $u \geq 0$ such that $s = \sup \kappa^{-1}(\{u\})$, if and only if s is the right limit of a positive length interval $[\kappa^{-1}(u-), \kappa^{-1}(u))$, or there exists $u' \geq 0$ such that $\{s\} = \kappa^{-1}(\{u'\})$. If both happen, then κ jumps at s from $u = \kappa(s-)$ to $u' = \kappa(s)$, and s is the right end point of $[\kappa^{-1}(u-), \kappa^{-1}(u))$.

There are two cases to consider. In the first case we have $\kappa^{-1}(\{u\}) = [\kappa^{-1}(u-), \kappa^{-1}(u)]$, which includes the situation where $\kappa^{-1}(\{u\}) = \{s\}$ is a one-point set. The right continuity of κ at $s = \kappa^{-1}(u)$ implies existence of a sequence

$$\{u_n, n \geq 1\} \subset \mathrm{Im}\kappa \text{ such that } u_n \downarrow u. \tag{6.7}$$

In particular, for each n we may choose $s_n \in \kappa^{-1}(\{u_n\})$. Note that $(s_n)_{n \geq 1} \subset (s, \infty)$ is strictly decreasing. Recalling the definition of γ and its monotonicity, together with Lemma 6.7 and the right continuity of g, we have

$$g(u) = \gamma(s) \leq \lim_{n \to \infty} \gamma(s_n) = \lim_{n \to \infty} g \tilde{\circ} \kappa(s_n) \leq \lim_{n \to \infty} g(u_n) = g(u).$$

Due to the already established monotonicity of γ, the equality $\gamma(s) = \lim_{n \to \infty} \gamma(s_n)$ is sufficient for concluding that γ is right continuous at s.

In the second case we have $\kappa^{-1}(\{u\}) = [\kappa^{-1}(u-), \kappa^{-1}(u))$. In particular, $s = \kappa^{-1}(u)$ but $\kappa(s) = u' > u = \kappa(s-)$, so (6.7) is false. Recalling the definition of γ and the assumption (H2), we have

$$\gamma(s) = g(u) = g(\kappa(s-)) = g(\kappa(s)-) = g(u'-).$$

Thus, if $[\kappa^{-1}(u'-), \kappa^{-1}(u'))$ has positive length, we obtain the (right-)continuity of γ at s from the reasoning of Remark 6.9. Otherwise, $\kappa^{-1}(\{u'\})$ is a single point. This implies (due to (H1)) that $g(u'-) = g(u')$ and moreover that $\gamma(s) = g \circ \kappa(s) = g(u') = g(u'-)$. In this case we can again use (6.7) with u replaced by u', and the subsequent reasoning to obtain the right continuity of γ at s. □

Lemma 6.12. *Let g, κ be as in Lemmas 6.10 and 6.11. Then $\gamma = g \tilde{\circ} \kappa \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ is a continuous function.*

Proof. Both $g(0) = 0$ and $\kappa(0) = 0$ so $g \circ \kappa(0) = g \tilde{\circ} \kappa(0) = \gamma(0) = 0$. Lemma 6.7 directly implies that $g \tilde{\circ} \kappa$ is both unbounded and strictly increasing at 0. Lemma 6.11 gives monotonicity. Therefore $\gamma \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$.

We next fix $s > 0$ and show that γ is left-continuous at s. Set $u := \kappa(s)$ and note that if s belongs to the interior of $\kappa^{-1}(\{u\})$, which equals $(\kappa^{-1}(u-), \kappa^{-1}(u))$ if non-empty, then the left-continuity of γ at s follows directly from the construction of $\tilde{\circ}$. Moreover, if $(\kappa^{-1}(u-), \kappa^{-1}(u))$ is non-empty then γ is left-continuous at $s = \kappa^{-1}(u)$ due to the same observation.

We now assume that $s = \kappa^{-1}(u-) \leq \kappa^{-1}(u)$ and consider a sequence $(s_n)_n$ of positive real numbers, which strictly increases to s. Our goal is to show that $\lim_n \gamma(s_n) = \gamma(s)$, as this together with the monontonicity of γ will imply left-continuity at s. We set $u_n := \kappa(s_n)$, $n \geq 1$, and $\bar{u} := \lim_{n\to\infty} u_n$. Then note that $\kappa(s-) = \bar{u} \leq u = \kappa(s)$ and $\gamma(s) = g(u-)$ for our choice of s. Using the definition and monotonicity of γ we have

$$g(u_n-) \leq \gamma(s_n) \leq \gamma(s).$$

Passing to the limit as $n \to \infty$ we obtain

$$g(\bar{u}-) \leq \gamma(s-) \leq \gamma(s) = g(u-). \tag{6.8}$$

If $\bar{u} = u$ then clearly $\gamma(s) = \gamma(s-) = g(u-)$. Otherwise, if $\bar{u} < u$, then (H2) implies

$$g(u-) = g(\kappa(s)-) = g(\kappa(s-)) = g(\bar{u}).$$

Consequently, if g is continuous at $\bar{u}$, then again $\gamma(s-) = \gamma(s)$ due to (6.8). Otherwise, $g(\bar{u}-) < g(\bar{u})$ and therefore $\kappa^{-1}(\{\bar{u}\})$ has positive length due to (H1). Since here $s = \kappa^{-1}(u-) = \kappa^{-1}(\bar{u})$, we have already covered this case above. This completes the proof of the lemma. □

Remark 6.13. The exact expression (formula) for the linear spline on conveniently chosen intervals $[\kappa^{-1}(u-), \kappa^{-1}(u)]$ was not important for the arguments above. The linear interpolation is the simplest, and it is compatible with taking inverses. However another continuous or differentiable increasing interpolation with compatible boundary values would equally imply an analogue of Lemma 6.12. Nevertheless, the following additivity result, necessary in the proof of an important property (P3) in Lemma 6.16, requires the spline to be linear.

The following lemma is an easy consequence of our choice of linear spline. We leave the details to the reader.

Lemma 6.14. *Let $h_1, h_2, \kappa \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$ be such that both (h_1, κ) and (h_2, κ) satisfy (H1) and (H2). Then $(h_1 + h_2, \kappa)$ also satisfies (H1) and (H2), and moreover*

$$h_1 \tilde{\circ} \kappa + h_2 \tilde{\circ} \kappa = (h_1 + h_2) \tilde{\circ} \kappa.$$

6.2 Solving for $\mathbf{T}(y)$ – the general case

Inspired by the analysis of the previous two sections, we now derive the general expression for the minimal solution to (5.2) in terms of a solution to a 1-dimensional optimization problem. It turns out that $\circ$ can be replaced with $\tilde{\circ}$ in (5.19), but arguing this rigorously is not as simple as one might guess.

Lemma 6.15. *Let $g_i \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$, $i \in [m]$, and f, κ be defined by (5.16) using inverses. Then for each $i \in [m]$ both (H1) and (H2) are satisfied for (g_i^{-1}, κ).*

Proof. Fix some $i \in [m]$. Let us check (H1). Suppose that $g_i^{-1}(u-) < g_i^{-1}(u)$ for some $u > 0$. Then in particular $f(u-) < f(u)$. Since κ is the (generalized right-continuous) inverse of f, it is also true that $f = \kappa^{-1}$ and in particular that

$$\kappa(s) = u, \quad \text{for all } s \in [f(u-), f(u)) = [\kappa^{-1}(u-), \kappa^{-1}(u)). \tag{6.9}$$

Let us now check (H2). Suppose that $\kappa(s-) < \kappa(s)$ for some $s > 0$. Then it must be (see also Lemma 6.1 iii)) that $f(u) = s$ for all $u \in [\kappa(s-), \kappa(s)) = [f^{-1}(s-), f^{-1}(s))$. Recalling that g_i^{-1}, $i \in [m]$, are all non-decreasing, their sum being constant on any interval implies that each of them is constant on the same interval. So $g_i^{-1}(u) = g_i^{-1}(f^{-1}(s-)) = g_i^{-1}(\kappa(s-))$ for all $u \in [\kappa(s-), \kappa(s))$, and this is equivalent to (H2) for (g_i^{-1}, κ). □

Recall that all the inverses are considered to be generalized right-continuous inverses. Let $g_i \in D_0^{\uparrow\uparrow}(\mathbb{R}_+)$, $i \in [m]$, and let κ be defined by (5.16). Furthermore, define

$$\gamma_i := g_i^{-1} \tilde{\circ} \kappa, \tag{6.10}$$

and $s \mapsto \vec{\gamma}(s)$ by

$$\vec{\gamma}(s) := (\gamma_1(s), \dots, \gamma_m(s)), \quad s \geq 0. \tag{6.11}$$

Note that the next two results are stated and proved in greater generality, although we will apply them only in the setting where g_i are given by (5.12).

Lemma 6.16. *(P1) The curve $\vec{\gamma}$ is a continuous curve in $\mathbb{R}^m$,*

(P2) for each i, γ_i is non-decreasing,

(P3) $s \mapsto \|\vec{\gamma}(s)\|_1$ is the identity map on $\mathbb{R}_+$,

(P4) for each $s \geq 0$

$$\kappa(s) \in \bigcap_{i=1}^{m} [g_i(\gamma_i(s)-), g_i(\gamma_i(s))] .$$

In particular, if $s \geq 0$ is such that g_i is continuous at $\gamma_i(s)$ for each $i \in [m]$, then

$$g_i(\gamma_i(s)) = g_{i+1}(\gamma_{i+1}(s))(= \kappa(s)), \ \forall i \in [m-1].$$

Proof. Properties (P1) and (P2) are both clearly satisfied due to Lemmas 6.11, 6.12 and 6.15.

Let us show (P3). Recalling that $f = \sum_{i=1}^m g_i^{-1}$, we have

$$\|\vec{\gamma}(s)\|_1 = \sum_{i=1}^{m} \left(g_i^{-1} \tilde{\circ} \kappa\right)(s) = (f \tilde{\circ} \kappa)(s) = s, \quad s \geq 0,$$

where the second identity is due to Lemma 6.14, and the third one is due to $f = \kappa^{-1}$ (see (6.2) and Remark 6.13).

Let us next verify (P4). Fix some $i \in [m]$ and recall that, due to the construction of $\tilde{\circ}$, if $\kappa(s) = u$ is not a jump point of g_i^{-1} then

$$\gamma_i(s) = g_i^{-1} \circ \kappa(s) = g_i^{-1}(u) = g_i^{-1}(u-).$$

Applying g_i to the above identity to get that $g_i(\gamma_i(s)) = g_i(g_i^{-1} \circ \kappa(s)) \geq \kappa(s)$, and also that (see also Lemma 6.1 part iii)) $g_i(\gamma_i(s)-) = g_i(g_i^{-1}(u)-) \leq u = \kappa(s)$.

Alternatively, if $\kappa(s) = u$ is a jump point of g_i^{-1}, then $s \in [\kappa^{-1}(u-), \kappa^{-1}(u)] = [f(u-), f(u)]$, and $\gamma_i = g_i^{-1} \tilde{\circ} \kappa$ is specified at s by linearly interpolating between $(f(u-), g_i^{-1}(u-))$ and $(f(u), g_i^{-1}(u))$. Therefore $g_i^{-1}(u-) \leq \gamma_i(s) \leq g_i^{-1}(u)$, We can now use the monotonicity of g_i, together with the RHS (resp. LHS) inequality to get

$$g_i(\gamma_i(s)-) \leq g_i(g_i^{-1}(u)-) \leq u = \kappa(s),$$
$$\big(\text{resp. } \kappa(s) = u \leq g_i(g_i^{-1}(u-)) \leq g_i(\gamma_i(s))\big),$$

where we used Lemma 6.1 iii) for both estimates. □

Proposition 6.17. *Let $\vec{g}, f, \kappa, \vec{\gamma}$ be as in Lemma 6.16, and suppose that we are given some $\vec{t} = (t_1, \dots, t_m) \in \mathbb{R}_+^m$ such that $g_i(t_i) = g_{i+1}(t_{i+1})$, $i \in [m-1]$. Then*

$$\vec{t} = \vec{\gamma}(\|\vec{t}\|_1),$$

provided that

(a) for each $i \in [m]$, g_i is continuous at t_i and strictly increasing from the left at t_i, or

(b) for each $i \in [m]$, g_i is strictly increasing from the right at t_i.

Proof. Let $\vec{t}$ be given as stated. Define $s := \|\vec{t}\|_1$ and $\vec{t'} = (t'_1, \dots, t'_m) := \vec{\gamma}(s)$. Our goal is to prove that $\vec{t'} = \vec{t}$ if (a) or (b) (or both).

Note initially that (P3) implies that $\|\vec{t'}\|_1 = s = \|\vec{t}\|_1$. So if $\vec{t} \neq \vec{t'}$ then there must exist some (minimal) index i such that both $t_i > t'_i$ and $t_{i+1} \leq t'_{i+1}$, otherwise $t_m > t'_m$ and $t_1 \leq t'_1$. Assuming (a), we would get from (P4) and monotonicity of g_i that

$$g_i(t_i) > g_i(t'_i) \geq \kappa(s) \geq g_{i+1}(t'_{i+1}-) \geq g_{i+1}(t_{i+1}-) = g_{i+1}(t_{i+1}) = g_i(t_i),$$

a contradiction. For the same reason as above, $\vec{t} \neq \vec{t'}$ implies that there must exist some (minimal) index j such that both $t_j < t'_j$ and $t_{j+1} \geq t'_{j+1}$, otherwise $t_m < t'_m$ and $t_1 \geq t'_1$. Assuming (b), we would get from (P4) and monotonicity of g_j that

$$g_j(t_j) < g_j(t'_j-) \leq \kappa(s) \leq g_{j+1}(t'_{j+1}) \leq g_{j+1}(t_{j+1}) = g_j(t_j),$$

which is again impossible.
If the the discrepancy is at m and 1 instead of at i and $i+1$ (resp. j and $j+1$), we would analogously arrive to a contradiction in the above argument under assumption (a) (resp. (b)). □

Recall that $\mathbf{T}(y) = \mathbf{T}^{\vec{\rho}}(\mathbb{x}; y)$ is the minimal solution of (5.2). Equivalently, $\mathbf{T}(y)$ is the solution of (5.13) and $(\mathbf{T}(y), s(y))$ (where $s(y)$ is defined in (5.17)) is the solution of (5.15).

The following theorem establishes Theorem 2.8(2).

Theorem 6.18. *Let $\vec{\gamma}$ be the continuous curve defined in (6.10), where g_i, $i \in [m]$, are given by (5.12). Then*

$$\vec{\gamma}(\|\mathbf{T}(y)\|_1) = \mathbf{T}(y), \quad \forall y \geq 0 \qquad \textit{s.t. } \mathbf{T}(y) \in \mathbb{R}_+^m.$$

Proof. Each $x_{i,j}$ is a *rcll* function, so it can have at most countably many discontinuities (and they are all jumps). On the set C defined by

$$C := \bigcap_{i,j=1}^{m} \{u \geq 0 : \ x_{i,j}(u-) = x_{i,j}(u)\} = \bigcap_{l=1}^{m} \{u \geq 0 : x_{*,l}(u-) = x_{*,l}(u)\}$$

clearly all $\underline{x}_i$ and (therefore) all g_i are (left-)continuous. The complement of C is either a finite (possibly empty) or a countable subset of $\mathbb{R}_+$.

Recall that $y \mapsto \mathbf{T}(y)$ is (component-wise) strictly increasing (and left-continuous, see Lemma 5.3). Therefore, $T_i^{-1}(C^c) := \{y \geq 0 : \ T_i(y) \in C^c\}$ is either a finite (possibly empty) or a countably infinite set, for each i. Define

$$Y := \bigcap_{i=1}^{m} T_i^{-1}(C) = \bigcap_{i=1}^{m} \{y \geq 0 : \ T_i(y) \in C\} \subset [0, \infty). \tag{6.12}$$

The above considerations imply that $Y^c = \bigcup_{i=1}^m T_i^{-1}(C^c)$ is a subset of a countable set. In particular, Y is dense in $[0,\infty)$.

We already know that $\mathbf{T}(y)$ satisfies (5.13) for any $y \geq 0$. Furthermore, if $y \in Y$, then the identities (A1) are fulfilled at $\vec{t} = \mathbf{T}(y)$. In other words, $g_i(t_i-) = g_i(t_i)$ for each i, and in particular $g_i(t_i) = g_{i+1}(t_{i+1})$, for all $i \in [m-1]$.

In addition, observe that Lemma 5.2 ii) and the definition of g_i in (5.12) imply that g_i is strictly increasing from the left at $t_i = T_i(y)$ for each $i \in [m]$, so that the additional hypotheses (a) of Proposition 6.17 is satisfied at $\mathbf{T}(y)$ for each $y \in Y$. Applying Proposition 6.17 for each $y \in Y$ separately, we conclude that

$$\vec{\gamma}(\|\mathbf{T}(y)\|_1) = \vec{\gamma}(s(y)) = \mathbf{T}(y), \quad \forall y \in Y. \tag{6.13}$$

Lemmas 5.3 and 6.12, joint with the fact that Y is dense in $\mathbb{R}_+$, now imply the stated claim. □

Recall that here and above $\mathbb{x}$ and $\vec{\rho}$ satisfy (2.15), and that $y \mapsto \mathbf{T}(y)$ depends on $\mathbb{x}$ and $\vec{\rho}$, while the map $s \mapsto \vec{\gamma}(s)$ is determined by $\mathbb{x}$. Our next goal is to prove Theorem 2.8(3) which includes the hypothesis that $\mathbf{T}(y) \in \mathbb{R}_+^m$. We therefore fix a y such that $\mathbf{T}(y) \in \mathbb{R}_+^m$. This simplifies our analysis of the optimization problem (5.15).

Indeed, we can now proceed in a way analogous to that in Section 5.3, relying on the powerful Theorem 6.18. We can now append an additional condition $\vec{t} = \vec{\gamma}(\|\vec{t}\|_1)$ to our optimization problem (5.15), with any given $j \in [m]$ as the reference index. The new and equivalent optimization problem is

$$\begin{cases} \vec{t} = \vec{\gamma}(s), \\ s = \|\vec{t}\|_1 = \|\vec{\gamma}(s)\|_1, \\ x_j(\vec{\gamma}(s)-) = -\rho_j y, \\ s \to \min. \end{cases} \tag{6.14}$$

The first two lines in (6.14) rely on (P3) and Theorem 6.18, the third line comes from (5.2), and the final line specifies the optimization rule. After solving for $s(y) \equiv s(y, \mathbb{x}, \rho)$, we will use it to find

$$\mathbf{T}(y) = \vec{\gamma}(s(y)). \tag{6.15}$$

It is important to note that $s(y)$ solves simultaneously each and every optimization problem

$$\begin{cases} x_j(\vec{\gamma}(s)-) = -\rho_j y, \\ s \to \min, \end{cases}, \quad j \in [m],$$

in complete analogy to (5.19)–(5.20). The above can be rewritten as $\mathbf{T}(y) = \vec{\gamma}(s(y))$, where for all $y \in \mathbb{R}_+$

$$s(y) = \min\{s \geq 0 : x_j(\vec{\gamma}(s)-) = -\rho_j y\}, \quad \forall j \in [m]. \tag{6.16}$$

We note that $y \mapsto s(y)$ is a left-continuous non-decreasing function. Indeed, x_j can be replaced in (6.16) with $\underline{x}_j$ from (5.5), and since $\underline{x}_j$ is non-increasing and continuous, we have that

$$s(y) = \min\left\{s \geq 0 : -\frac{1}{\rho_j}\underline{x}_j(\vec{\gamma}(s)) \geq y\right\} = \inf\left\{s \geq 0 : -\frac{1}{\rho_j}\underline{x}_j(\vec{\gamma}(s)) \geq y\right\}, \quad y \geq 0,$$

so that s is the left-continuous generalised inverse of $-\frac{1}{\rho_j}\underline{x}_j \circ \vec{\gamma}$. Furthermore, if $y \in Y$ (where Y is the "good set" from the proof of Theorem 6.18) then it is easy to see that $s(y) = \tilde{s}(y)$ where

$$\tilde{s}(y) := \min\{s \geq 0 : x_j(\vec{\gamma}(s)) = -\rho_j y\}, \quad \forall j \in [m]. \tag{6.17}$$

Indeed, the condition in (6.16) is satisfied earlier than the condition in (6.17), so that $s(y) \leq \tilde{s}(y)$ for all y. However, if $y \in Y$, then $s(y)$ also solves (6.17), implying the reversed inequality.

Now define for each $i \in [m]$

$$S_i(y) = \inf\left\{s \geq 0 : \, x_i \circ \vec{\gamma}(s-) = \lim_{u \uparrow s} x_i \circ \gamma(u) = -\rho_i y\right\}, \quad y \geq 0. \tag{6.18}$$

Note that $s \mapsto x_i \circ \vec{\gamma}(s)$ is again a rcll (or càdlàg) function on $[0, +\infty)$, with no negative jumps. Therefore, when $S_i(y) < \infty$, $\inf$ could be replaced by $\min$. Moreover, it is easy to see that S_i is again a left-continuous and non-decreasing function.

Corollary 6.19. *The maps $y \mapsto S_i(y)$ and $y \mapsto s(y)$ are identical for each $i \in [m]$. In particular, whenever $\mathbf{T}(y+) \in \mathbb{R}^m_+$ then $S_i(y+) < \infty$ and*

$$S_i(y+) - S_i(y) = \|\mathbf{T}(y+) - \mathbf{T}(y)\|_1$$

for all $i \in [m]$.

Proof. For the first part of the statement, it is enough to show that s and S_i coincide on the dense set Y, because both S_i and s are left-continuous functions.

Comparing the conditions in (6.17) and in (6.18), it is clear that $S_i(y) \leq \tilde{s}(y)$ for any y. However if $y \in Y$, then $s(y) = \tilde{s}(y)$ implying

$$S_i(y) \leq s(y), \; y \in Y.$$

To prove the reversed inequality, we use the fact that x_i (and therefore $x_i \circ \vec{\gamma}$) has no non-negative jumps for each $i \in [m]$ and the monotonicity and continuity of $\vec{\gamma}$. More precisely, $\vec{\gamma}$ can be either strictly increasing from the left at r, or constant on some interval $(r - \varepsilon, r]$ of positive length. In the former case, $x_i \circ \vec{\gamma}(r-)$ equals $x_i \circ \vec{(}\gamma(r)-)$, while in the latter case $x_i \circ \vec{\gamma}(r-) = x_i \circ \vec{\gamma}(r) \geq x_i(\vec{\gamma}(r)-)$. The hereby verified inequality

$$x_i \circ \vec{\gamma}(r-) \geq x_i(\vec{\gamma}(r)-), \; r \geq 0,$$

implies that the condition in (6.16) is satisfied earlier than that in (6.18), therefore

$$s(y) \leq S_i(y), \; y \in [0, \infty),$$

which concludes the argument for $s \equiv S_i$.

The second part of the statement directly follows from the (strict) monotonicity of $y \mapsto \mathbf{T}(y)$ yielding $\|\mathbf{T}(y+) - \mathbf{T}(y)\|_1 = \|\mathbf{T}(y+)\|_1 - \|\mathbf{T}(y)\|_1$. □

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