Ill-posedness of the pure-noise Dean–Kawasaki equation[*][†]

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Abstract

We prove that the Dean–Kawasaki-type stochastic partial differential equation

$$\partial\rho = \nabla\cdot(\sqrt{\rho}\,\xi) + \nabla\cdot\big(\rho\, H(\rho)\big)\ ,$$

with vector-valued space-time white noise ξ, does not admit solutions for any initial measure and any vector-valued bounded measurable function H on the space of measures. This applies in particular to the pure-noise Dean–Kawasaki equation ($H\equiv 0$). The result is sharp, in the sense that solutions are known to exist for some unbounded H.

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1 Introduction and the main result

Let $\mathbb{M}^d$ be either the standard d-dimensional Euclidean space $\mathbb{R}^d$ or the flat d-dimensional torus $\mathbb{T}^d$, $d\geq 1$. For $k\in\mathbb{N}_0$, we let $\mathcal{C}_b^k$ be the space of all continuous and bounded real-valued functions on $\mathbb{M}^d$ with continuous and bounded derivatives up to order k, and we set $\mathcal{C}_b \coloneqq \mathcal{C}_b^0$, endowed with the uniform norm $\|\cdot\|_0$. For a Borel measure μ on $\mathbb{M}^d$ and a Borel function $f\colon \mathbb{M}^d\to\mathbb{R}$, we write $\mu f\coloneqq \int f\,\mathrm{d}\mu$ whenever the integral makes sense. We denote by $\mathscr{M}_b^+$ the space of all positive finite Borel measures on $\mathbb{M}^d$, endowed with the narrow topology, i.e. the coarsest topology for which all the functionals $\mu\mapsto\mu f$, with $f\in\mathcal{C}_b$, are continuous.

On $\mathbb{M}^d$ we consider the *Dean–Kawasaki equation*

$$\mathrm{d}\mu_t = \alpha\,\Delta\mu_t\,\mathrm{d}t + G(\mu_t)\,\mathrm{d}t + \nabla\cdot(\sqrt{\mu_t}\,\xi)\ , \tag{1.1}$$

where $\alpha\geq 0$ is a parameter, $G\colon \mathscr{M}_b^+\to\mathbb{R}$ is Borel measurable, ξ is an $\mathbb{R}^d$-valued space-time white noise, and $(\mu_t)_{t\geq 0}$ is an $\mathscr{M}_b^+$-valued stochastic process with a.s. continuous paths.

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The equation with $\alpha > 0$ has been proposed by K. Kawasaki in [20] and, independently, by D. S. Dean in [8], to describe the density function of a system of $N \gg 1$ particles subject to a diffusive Langevin dynamics, with the noise ξ describing the particles' thermal fluctuations. Equations like (1.1) — possibly with a different non-linearity in the noise term — fall within the class of Ginzburg–Landau stochastic phase field models, and effectively describe super-cooled liquids, colloidal suspensions, the glass-liquid transition, bacterial patterns, and other systems; see, e.g., the recent review [28].

From a mathematical point of view, these equations model in the continuum the *fluctuating hydrodynamic theory* of interacting particle systems; see, e.g., [27, 11, 2, 14] and the review [3]. A specific interest in the case of (1.1) — i.e., with a square-root non-linearity in the noise term — is partially motivated by the structure of the noise in connection with the geometry of the L^2-Kantorovich–Rubinstein–Wasserstein space $(\mathscr{P}_2, W_2)$. Indeed, in the free case ($H \equiv 0$), a solution μ_t to (1.1) with $\alpha = 1$ is an intrinsic random perturbation of the gradient flow of the Boltzmann–Shannon entropy on $\mathscr{P}_2$ by a noise ξ distributed according to the energy dissipated by the system, i.e. by the natural isotropic noise arising from the Riemannian structure of $\mathscr{P}_2$, see [18, 25, 24, 1].

We would like to stress that we consider here Dean–Kawasaki-type equations with *white* noise: a very fruitful theory has been developed for similar equations with *colored*, *truncated*, or otherwise *approximated* noise (both of Itô and Stratonovich type), abstractly [6, 14, 15, 16, 17], numerically [5, 7, 12], and — for both colored and white noise — approaching concrete applications [11, 13].

1.1 Main result

A rigorous definition of solutions to (1.1) was introduced by V. Konarovskyi, T. Lehmann, and M.-K. von Renesse in [21] for $G \equiv 0$, and in [22] when

$$G(\mu) = \nabla \cdot \big(\mu\, H(\mu)\big) \tag{1.2}$$

for $H \colon \mathscr{M}_b^+ \to \mathbb{R}^d$, as we now recall.

Definition 1.1 (Martingale solutions, cf. [22, Dfn. 1])**.** *Fix $T \in (0, \infty)$ and let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space. A continuous $\mathscr{M}_b^+$-valued process $\mu_\bullet \coloneqq (\mu_t)_{t\in[0,T]}$ on $(\Omega, \mathscr{F})$ is a* solution to (1.1) *(*up to time T*) if, for each $f \in \mathcal{C}_b^2$ the process $M_\bullet^f \coloneqq \big(M_t^f\big)_{t\in[0,T]}$ with*

$$M_t^f \coloneqq \mu_t f - \mu_0 f - \int_0^t \mu_s\big(\tfrac{\alpha}{2}\Delta f + \nabla f \cdot H(\mu_s)\big)\,\mathrm{d}s\,, \qquad t \in [0,T]\,,$$

is a continuous $\mathbf{P}$-martingale on $(\Omega, \mathscr{F})$ with respect to the filtration $\mathscr{F}_\bullet \coloneqq (\mathscr{F}_t)_{t\in[0,T]}$ generated by $\mu_\bullet$, with quadratic variation

$$\big[M^f\big]_t = \int_0^t \mu_s\, |\nabla f|^2\,\mathrm{d}s\,, \qquad t \in [0,T]\,.$$

In the case when $\alpha > 0$ and $H(\mu) = \nabla \frac{\delta F(\mu)}{\delta \mu}$ for some sufficiently smooth and *bounded* $F \colon \mathscr{M}_b^+ \to \mathbb{R}$, Konarovskyi, Lehmann, and von Renesse have shown in [21, 22] that (1.1) admits solutions if and only if the initial datum μ_0 is the empirical measure of a finite particle system, i.e. μ_0 is a purely atomic measure and each atom has mass $1/\alpha$. In this case, the solution $\mu_\bullet$ exists for all times, is unique and identical with the empirical measure of the Langevin particle systems with mean-field interaction F. Further extensions of these rigidity results were subsequently obtained by Konarovskyi and Müller in [23] and by Müller, von Renesse, and Zimmer in [26].

Their technique, however, does not apply to the case $\alpha = 0$, hence in particular it does not cover the pure-noise Dean–Kawasaki equation. Here, we complete the picture by addressing precisely this case.

Theorem 1.2. *Let $\alpha = 0$ and $G(\mu) = \nabla \cdot \big(\mu\, H(\mu)\big)$ for some bounded Borel $H \colon \mathscr{M}_b^+ \to \mathbb{R}^d$. Then* (1.1) *has no solutions for any initial condition $\mu_0 \in \mathscr{M}_b^+$.*

This result is sharp, in the sense that existence of solutions was shown by Konarovskyi and von Renesse in [24, 25] for the Dean–Kawasaki equation on the real line with singular drift

$$\mathrm{d}\mu_t = \sum_{x : \mu_t\{x\}>0} \Delta \delta_x \, \mathrm{d}t + \nabla \cdot \left(\sqrt{\mu_t}\; \xi\right) , \tag{1.3}$$

that is, in the case when H in (1.2) is *un*bounded. Existence of solutions to (1.3) were eventually constructed by the first named author also on compact manifolds [9] and in other more general settings [10].

2 Proofs

For any real-valued function f we denote by Σ_f the *singular set* of f, i.e. the set of points in the domain of f at which f is *not* differentiable. If not stated otherwise, $(\Omega, \mathscr{F}, \mathbf{P})$ is a complete probability space, and we denote by $\mathbf{E}$ the $\mathbf{P}$-expectation. Further let $\mu_\bullet$ be a solution to (1.1) up to time T on $(\Omega, \mathscr{F}, \mathbf{P})$ and assume that

$$\mathbf{E}[\mu_0 \mathbb{M}^d] < \infty \; . \tag{2.1}$$

(Note that (2.1) is trivially satisfied if μ_0 is deterministic.)

We start with some preparatory lemmas.

Lemma 2.1. *If $\mu_\bullet$ is a solution to* (1.1) *up to time T, then $\mu_t \mathbb{M}^d = \mu_0 \mathbb{M}^d$ a.s. for all $t \in [0,T]$.*

Proof. Choosing $f = \mathbf{1}$ in the martingale problem in Definition 1.1, we have $\left[M^{\mathbf{1}}\right]_t = 0$ for all times. It follows that $\mu_t \mathbb{M}^d = M_t^{\mathbf{1}}$ is a.s. a constant martingale, and therefore $\mu_t \mathbb{M}^d = \mu_0 \mathbb{M}^d$ a.s. for all times. ∎

For each $t > 0$ define a measure μ_t^* on $\mathbb{M}^1$ as

$$\mu_t^* \coloneqq \mathbf{E} \int_0^t \int_{\mathbb{M}^{d-1}} \mu_s(\,\cdot\,, \mathrm{d}x_2, \ldots, \mathrm{d}x_d)\, \mathrm{d}s \tag{2.2}$$

Whenever the assumption in (2.1) is satisfied, μ_t^* is a finite measure by Lemma 2.1, hence the set A_t of its atoms is at most countable.

Throughout the rest of this work we assume that (2.1) holds, we fix $T > 0$ and we set $A \coloneqq A_T$.

Lemma 2.2. *There exists a continuous function $g \colon \mathbb{M}^1 \to \mathbb{R}$ with the following properties:*

(i) *g is piecewise affine, non-negative, and bounded;*

(ii) *$\Sigma_g \cap A = \varnothing$;*

(iii) *Σ_g is at most countable and $|g'| = 1$ on Σ_g^{c};*

Proof. We may dispense with showing that g is non-negative. Indeed, suppose we have found some function g with all the required properties except non-negativity. Then, $g - \inf g$ still satisfies all these properties, since it has the same singular set as g, and is additionally non-negative.

Assume $\mathbb{M}^1 = \mathbb{R}$. Fix $y_0 \in A^c$, and define inductively a countable set $Y := \{y_k\}_{k\in\mathbb{Z}}$ in the following way: if $k \in \mathbb{Z}^\pm$, choose $y_k \in A^c$ such that $|y_k - (y_{k\mp 1} \pm 1)| \le 2^{-k}$. Further set

$$a_k := \begin{cases} \sum_{i=1}^{k} (-1)^{i-1}(y_i - y_{i-1}) & \text{if } k \in \mathbb{Z}^+ \ , \\ 0 & \text{if } k = 0 \ , \\ \sum_{i=k}^{-1} (-1)^{i+1}(y_{i+1} - y_i) & \text{if } k \in \mathbb{Z}^- \ . \end{cases}$$

In this way, $Y \subset A^c$ and $|a_k| \le 2$ for every $k \in \mathbb{Z}$. It follows that the linear spline g interpolating the points $((y_k, a_k))_{k\in\mathbb{Z}}$ has all the desired properties (with the possible exception of non-negativity) and in particular satisfies $\|g\|_0 \le \sup_k |a_k| \le 2$.

Assume $\mathbb{M}^1 = \mathbb{T}^1$. All sets and points in the rest of the proof are regarded $\mod 1$. Since A is countable, $A_1 := A \cup (A + \{1/2\})$ is countable too, and we can choose $y \notin A_1$, which implies that $y + 1/2 \notin A_1$ as well. Then, the function g defined as the piecewise affine function with singular set $\{y, y + 1/2\}$ and interpolating the points $(y, 0)$ and $(y + 1/2, 1/2)$ has all the desired properties. ■

Proposition 2.3. *Fix $T \in (0,\infty)$, and let $\mu_\bullet := (\mu_t)_{t\le T}$ be a solution — if any exists — to* (1.1) *for $\alpha = 0$ up to time T. Further suppose that: $f_n \colon \mathbb{M}^d \to \mathbb{R}$ is a function in $\mathcal{C}^2_b$ for each $n \in \mathbb{N}$, $f \colon \mathbb{M}^d \to \mathbb{R}$ is a function in $\mathcal{C}^0_b$, $h \colon \mathbb{M}^d \to \mathbb{R}^d$ is a Borel measurable function with $h \equiv \nabla f$ on Σ_f^c, satisfying*

(a) $\lim_n f_n = f$ *uniformly on* $\mathbb{M}^d$*;*

(b) $\lim_n \int_0^T \mu_s |\nabla f_n - h| \,\mathrm{d}s = 0$ *a.s.*

Then, the process $M_\bullet := (M_t)_{t\in[0,T]}$ with

$$M_t := \mu_t f - \mu_0 f - \int_0^t \mu_s\big(h \cdot H(\mu_s)\big)\,\mathrm{d}s \ , \qquad t \in [0,T] \ , \tag{2.3}$$

is a martingale with respect to the filtration $\mathscr{F}_\bullet := (\mathscr{F}_t)_{t\in[0,T]}$ generated by $\mu_\bullet$, with quadratic variation

$$[M]_t = \int_0^t \mu_s |h|^2 \,\mathrm{d}s \ , \qquad t \in [0,T] \ . \tag{2.4}$$

Proof. By Definition 1.1, for every $n \in \mathbb{N}$, the processes $M^n_\bullet := (M^n_t)_{t\in[0,T]}$ with

$$M^n_t := \mu_t f_n - \mu_0 f_n - \int_0^t \mu_s\big(\nabla f_n \cdot H(\mu_s)\big)\,\mathrm{d}s \ , \qquad t \in [0,T] \ , \tag{2.5}$$

is a continuous martingale w.r.t. the *same* filtration $\mathscr{F}_\bullet$, with quadratic variation

$$[M^n]_t = \int_0^t \mu_s |\nabla f_n|^2 \,\mathrm{d}s \ , \qquad t \in [0,T] \ . \tag{2.6}$$

The conclusion will follow letting $n \to \infty$ in (2.5) and applying [4, Lem. B.11], provided we show that $M^n_\bullet$ converges to $M_\bullet$ in probability uniformly on $[0,T]$, that is

$$\mathbb{P}\text{-}\lim_n \sup_{t\le T} |M^n_t - M_t| = 0 \ . \tag{2.7}$$

We show the stronger statement that

$$\lim_n \sup_{t\leq T} |M_t^n - M_t| = 0 \quad \text{a.s.}$$

Indeed, by (a),

$$\lim_n |\mu_0 f_n - \mu_0 f| = 0 \quad \text{a.s.} \tag{2.8}$$

By Lemma 2.1 and by (a),

$$\lim_n \sup_{t\in[0,T]} |\mu_t f_n - \mu_t f| \leq \lim_n \sup_{t\in[0,T]} \mu_t \mathbb{M}^d \left\| f_n - f \right\|_0 = \mu_0 \mathbb{M}^d \lim_n \left\| f_n - f \right\| = 0 \; . \tag{2.9}$$

By Cauchy–Schwarz inequality, uniform boundedness of $H\colon \mathscr{M}_b^+ \to \mathbb{R}^d$, and (b),

$$\begin{aligned}\lim_n \sup_{t\in[0,T]} &\left| \int_0^t \mu_s\big(\nabla f_n \cdot H(\mu_s)\big)\,\mathrm{d}s - \int_0^t \mu_s\big(h \cdot H(\mu_s)\big)\,\mathrm{d}s \right| \leq \\ &\leq \lim_n \sup_{t\in[0,T]} \int_0^t \mu_s \left| (\nabla f_n - h)\cdot H(\mu_s) \right| \mathrm{d}s \\ &\leq \left\| H \right\|_0 \lim_n \int_0^T \mu_s \left| \nabla f_n - h \right| \mathrm{d}s = 0 \; . \end{aligned} \tag{2.10}$$

Combining (2.8), (2.9), and (2.10) shows (2.7) and thus the assertion. ■

We are now ready to prove our main result.

Proof of Theorem 1.2. Fix $\mu_0 \in \mathscr{M}_b^+$ and set $c \coloneqq \mu_0 \mathbb{M}^d > 0$. We argue by contradiction that there exists a solution $(\mu_t)_t$ to (1.1) starting at μ_0.

Let g be the function constructed in Lemma 2.2 and, for every $\varepsilon > 0$, define $g_\varepsilon\colon \mathbb{M}^1 \to \mathbb{R}$ as a regularization of g satisfying: (a_g) $g_\varepsilon \in \mathcal{C}_b^2$ and g_ε converges to g uniformly on $\mathbb{M}^1$ as $\varepsilon \downarrow 0$; (b_g) g'_ε converges to $g' \equiv 1$ locally uniformly away from Σ_g as $\varepsilon \downarrow 0$; (c_g) $|g'_\varepsilon| \leq 1$ everywhere on $\mathbb{M}^1$. Finally, define $f_\varepsilon\colon \mathbb{M}^d \to \mathbb{R}$ and $f\colon \mathbb{M}^d \to \mathbb{R}$ by $f_\varepsilon(x) \coloneqq g_\varepsilon(x_1)$ and $f(x) \coloneqq g(x_1)$ respectively, where $x = (x_1, \dots, x_d) \in \mathbb{M}^d$. Now, let $\varepsilon \coloneqq 1/n$ and put, for simplicity of notation, $f_n \coloneqq f_{\varepsilon_n}$ From (a_g)-(c_g) above we deduce the analogous properties for f_n and f, that is

(a_f) $f_n \in \mathcal{C}_b^2$ converges to f uniformly on $\mathbb{M}^d$ as $n \to \infty$;

(b_f) ∇f_n converges to ∇f locally uniformly away from Σ_f as $n \to \infty$;

(c_f) $|\nabla f_n| \leq 1$ everywhere on $\mathbb{M}^d$.

Step 1 We start by verifying the assumptions in Proposition 2.3. The singular set Σ_f of f satisfies $\Sigma_f = \Sigma_g \times \mathbb{M}^{d-1}$. Thus, for every $t \in [0,T]$,

$$\mathbf{E} \int_0^t \mu_s \Sigma_f \,\mathrm{d}s \leq \mathbf{E} \int_0^T \mu_s \Sigma_f \,\mathrm{d}s = \mathbf{E} \int_0^T \mu_s (\Sigma_g \times \mathbb{M}^{d-1}) \,\mathrm{d}s = \mu_T^* \Sigma_g = 0$$

by Lemma 2.2(ii), and therefore

$$\int_0^t \mu_s \Sigma_f \,\mathrm{d}s = 0 \quad \text{a.s.} \,, \qquad t \in [0,T] \; . \tag{2.11}$$

Respectively: by (2.11); since $(\nabla f)(x) = g'_\varepsilon(x_1) = 1$ on Σ_f^{c} by definition of f and Lemma 2.2(iii); and by Lemma 2.1,

$$\int_0^t \mu_s \left| \nabla f \right|^2 \mathrm{d}s = \int_0^t \mu_s\big|_{\Sigma_f^{\mathrm{c}}} \left| \nabla f \right|^2 \mathrm{d}s = \int_0^t \mu_s \mathbf{1} \,\mathrm{d}s = c\,t \quad \text{a.s.} \,, \qquad t \in [0,T] \,, \tag{2.12}$$

which shows in particular that the integral in the left-hand side of (2.12) is well-defined for every $t \in [0,T]$ and thus that

$$\mu_s \left|\nabla f\right|^2 = \mu_s \mathbf{1} \quad \text{is a.s. well-defined for a.e. } s \in [0,T] \;.$$

This shows that in Proposition 2.3 we may choose $h = \nabla f$.

Fix $s \in [0,T]$. Since μ_s is a.s. a finite measure, by the convergence in (b_f) and Dominated Convergence in $L^1(\mu_s)$ because of (c_f),

$$\lim_{n\to\infty} \int \left|\nabla f_n - \nabla f\right| \mathrm{d}\mu_s = 0 \qquad \text{a.s.} \;, \quad \text{for a.e. } s \in [0,T] \;. \tag{2.13}$$

By (c_f) and Lemma 2.1,

$$\mu_s \left|\nabla f_n - \nabla f\right| \leq 2\mu_s \mathbf{1} = 2c \qquad \text{a.s.} \;, \quad \text{for a.e. } s \in [0,T] \;, \quad n \in \mathbb{N} \;.$$

Thus, the function $s \mapsto \mu_s \left|\nabla f_\varepsilon - \nabla f\right|$ is a.s. $\mathscr{L}^1$-essentially bounded on $[0,T]$ uniformly in n. By the convergence in (2.13) for a.e. $s \in [0,T]$ and Dominated Convergence in $L^1([0,T])$ with dominating function $2c \in L^1([0,T])$,

$$\lim_{n\to\infty} \int_0^T \mu_s \left|\nabla f_n - \nabla f\right| \mathrm{d}s = 0 \qquad \text{a.s.} \tag{2.14}$$

Note that (a_f) verifies the assumption in Proposition 2.3(a), while (2.14) verifies Proposition 2.3(b).

Step 2 Applying Proposition 2.3 with f as above and $h \equiv \nabla f$, the process $B_\bullet \coloneqq (B_t)_{t\in[0,T]}$ with

$$B_t \coloneqq \mu_t f - \mu_0 f - \int_0^t \mu_s\big(\nabla f \cdot H(\mu_s)\big)\, \mathrm{d}s \;, \qquad t \in [0,T] \;,$$

is well-defined and a continuous martingale w.r.t. $\mathscr{F}_\bullet$ with quadratic variation

$$[B]_t = \int_0^t \mu_s \left|\nabla f\right|^2 \mathrm{d}s = c\, t \;, \qquad t \in [0,T] \;.$$

By Lévy's characterization, the process $W_\bullet \coloneqq (W_t)_{t\in[0,T]}$ with $W_t \coloneqq B_{t/c}$, is a standard one-dimensional Brownian motion. Note that $c \coloneqq \mu_0(\mathbb{M}^d) > 0$ is $\mathscr{F}_0$-measurable, therefore it is independent of $W_\bullet$ since the latter is an $\mathscr{F}_\bullet$-Brownian motion, see e.g. [19, Prob. 2.5.5, p. 73]. As a consequence, the set

$$E \coloneqq \{B_T < -c \left\|H\right\|_0 T\}$$

has positive $\mathbf{P}$-probability.

On the one hand, on the set of positive probability E,

$$\begin{aligned} \mu_T f &= \int_0^T \mu_s\big(\nabla f \cdot H(\mu_s)\big)\, \mathrm{d}s + B_T \\ &< \left\|H\right\|_0 \int_0^T \mu_s \mathbf{1}\, \mathrm{d}s - c \left\|H\right\|_0 T = 0 \;. \end{aligned}$$

On the other hand, $\mu_T f$ is a.s. non-negative, since f is a non-negative function by the choice of g and Lemma 2.2(i). Thus we have reached a contradiction, as desired. ∎

3 Possible extensions

Let us collect here some observations about possible extensions of our main result.

Solutions to the free Dean–Kawasaki equation have been constructed in [9, 10, 21] in a far more general setting than $\mathbb{M}^d$, encompassing e.g. Riemannian manifolds, as well as some 'non-smooth spaces'. For the sake of simplicity, let us discuss the case of a Riemannian manifold M with Riemannian metric g. A definition of solution to (1.1) is given again in terms of the martingale problem in Definition 1.1, replacing the Laplacian on $\mathbb{M}^d$ with the Laplace–Beltrami operator Δ_{g} on M, the gradient with the Riemannian gradient ∇^{g} induced by g, and the scalar product with the metric g itself.

We expect the non-existence result in Theorem 1.2 to be a *structural property* of the equation, rather than a feature of the ambient space, and thus to extend to this more general setting as well. Indeed, given a solution $\mu_\bullet$ up to time T, the proof depends only on the construction of a function $f\colon M \to \mathbb{R}^+$ satisfying $|\nabla f| \equiv \mathbf{1}$ on some Borel set $A \subset M$ μ_t-negligible for $\mathscr{L}^1$-a.e. $t \in [0,T]$. To control this negligibility when $M = \mathbb{M}^d$, we introduced the measure μ_T^* in (2.2) as the time average of the marginal of $\mu_\bullet$ on $\mathbb{M}^1$ with respect to the projection onto the *first coordinate*. On a general manifold, this can be done by choosing μ_T^* as the time average of the marginal of $\mu_\bullet$ on $\mathbb{R}_0^+$ with respect to the projection onto the *radial coordinate* in a spherical coordinate system centered at any point $o \in M$, viz.

$$\mu_T^*[0,r) := \int_0^T \mu_s B_r(o)\,\mathrm{d}s\;, \qquad T>0\;, \quad r>0\;,$$

where $B_r(o)$ is the ball in M of radius $r > 0$ and center o w.r.t. the intrinsic distance d_{g} on M induced by g. A function $g\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ may then be constructed from Lemma 2.2, so that $f(x) = g\big(\mathsf{d}_{\mathsf{g}}(x,o)\big)$ has the desired properties.

For a general manifold M, the argumentation above is not sufficient to prove the conclusion, since we also need to show that μ_s vanishes on the singular set $\Sigma_f \subset M$ of f, and this set includes the *cut locus* of the point o, which is generally 'large' and wildly dependent on o. However, the argument can be made rigorous on manifolds with only *one chart*, (including Euclidean spaces, hyperbolic spaces, etc.) in which the cut locus of any o is empty, and on standard spheres, in which the cut locus of o exactly consists of its antipodal point.

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